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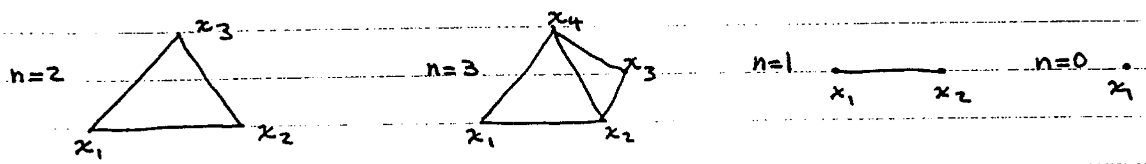
Suppose we have a triangulation in an n-dim space.

Then we have a Δ with $n+1$ vertices:

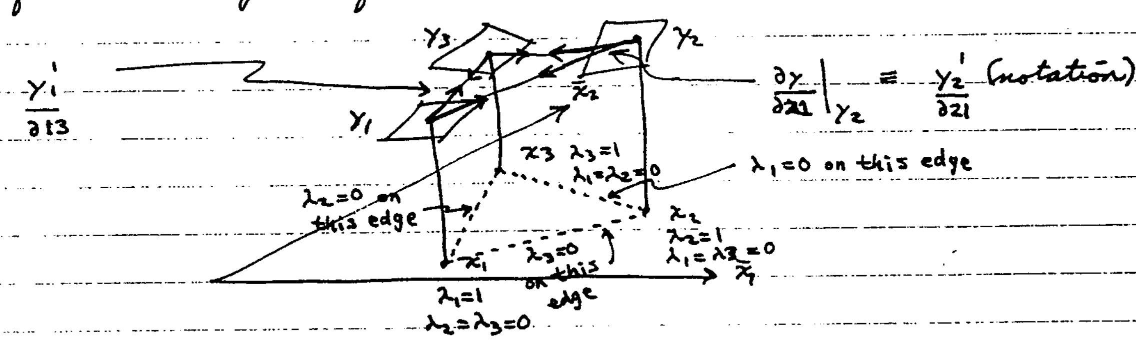


At each vertex we specify a value y and a multiple regression $\frac{\partial y}{\partial \bar{x}_i}$ where $\bar{x}_i$ are axes.

We determine the coordinates $\lambda_1, \dots, \lambda_{n+1}$ specifying how close we are to vertex x_i.
$\lambda_i = 1$ at vertex x_i. $\lambda_i = 0$ at vertex $x_{j \neq i}$.

$\sum_i \lambda_i = 1$. See p. 40 for calculation of λ's.

We transform our derivatives into λ coordinates: $\partial y / \partial \lambda_i$. These derivatives are in the directions of the edges of the Δ:



We want to match the values and the slopes at the vertices.

We have $n+1$ values for $n+1$ vertices plus n derivatives at each of $n+1$ vertices giving another $n(n+1)$ values.

Thus the total number of values is $(n+1)^2$.

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Then our splines have the following form:

$n=0$: $y = y_1$ trivial case since value at pt is value at pt

$n=1$: $$y = f_1(\lambda_1)\,y_1 + f_1(\lambda_2)\,y_2 + f_2(\lambda_1,\lambda_2)\,\frac{y_1'}{\partial 12} + f_2(\lambda_2,\lambda_1)\,\frac{y_2'}{\partial 21}$$

where $$f_1(\lambda) \equiv 3\lambda^2 - 2\lambda^3$$

$$f_2(\lambda_1,\lambda_2) \equiv \tfrac{2}{3}(\lambda_1^3 - \lambda_1) - \tfrac{1}{3}(\lambda_2^3 - \lambda_2)$$

□ Our notation is $\frac{y_1'}{\partial 12} \equiv \left.\frac{\partial y}{\partial \vec{v}}\right|_{x_1}$ where $\vec{v}$ is in direction from x_1 to x_2.

We observe that λ_1 decreases from 1 to 0 as we move from x_1 to x_2. Thus λ_1 is a natural coordinate for computing the directional derivatives. Furthermore, we have $y_2 = 1 - \lambda_1$. Thus, along the line from x_1 to x_2, we have

$$\frac{\partial y}{\partial \lambda_2} = -\frac{\partial y}{\partial \lambda_1} \quad \text{on line from } x_1 \text{ to } x_2.$$

In the general n-dimensional case, we find that on edges connecting vertices our λ's are always the natural coordinates.

If we have x_1, x_2, x_3 and we want $\frac{\partial y}{\partial 12}$,

then we observe that on the edge connecting x_1 and x_2 we have λ_1 decreasing from 1 to 0, $\lambda_2 = 1 - \lambda_1$, and $\lambda_3 = 0$.

Suppose we write $y = f(\lambda_1, \lambda_2, \lambda_3)$.

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Then we have that $\frac{\partial y}{\partial 12} \equiv \frac{\partial y}{\partial \vec{v}}$ ($\vec{v}$ from x_1 to x_2)

is really just $\frac{\partial f(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial \vec{v}}$.

$$\frac{\partial f(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_1} = \left(\frac{\partial f}{\partial \lambda_1} + \frac{\partial f}{\partial \lambda_2} \frac{\partial \lambda_2}{\partial \lambda_1} + \frac{\partial f}{\partial \lambda_3} \frac{\partial \lambda_3}{\partial \lambda_1} \right) \frac{\partial \lambda_1}{\partial \vec{v}}$$

$$= \left(\frac{\partial f}{\partial \lambda_1} + \frac{\partial f}{\partial \lambda_2}(-1) + \frac{\partial f}{\partial \lambda_3} \cdot 0 \right)(-1)$$

$$= -\frac{\partial f}{\partial \lambda_1} + \frac{\partial f}{\partial \lambda_2}$$

Thus the partials for λ's other than those on the x_1, x_2 edge are zero and may be ignored. Regardless of the number of dimensions in our space, only two partials are nonzero for any edge connecting two points.

□ Note that for $n=1$ we have a term $\frac{y_2'}{\partial 21}$.

This is the derivative in the direction from x_2 to x_1, the negative (if $x_2 > x_1$) of $\partial y / \partial x |_{x_2}$. The usual cubic spline

definition uses $\partial y / \partial x$. Our use of the new notation makes our definition symmetric and generalizes to higher dimensions.

□ We can validate our formula for $n=1$ by checking that $y|_{x_1} = y_1$ $y|_{x_2} = y_2$ $\left.\frac{\partial y}{\partial 12}\right|_{x_1} = \frac{y_1'}{\partial 12}$

$$\left.\frac{\partial y}{\partial 12}\right|_{x_2} = \frac{y_2'}{\partial 12}.$$

In others words, we must match specified values & slopes.

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$$y\Big|_{x_1} = y\Big|_{\lambda_1=1,\,\lambda_2=0} = f_1(1)\,y_1 + f_1(0)\,y_2 + f_2(1,0)\,\frac{y_1'}{\partial 12} + f_2(0,1)\,\frac{y_2'}{\partial 21}$$

$$f_1(0) = 3\cdot 0^2 - 2\cdot 0^3 = 0 - 0 = 0$$

$$f_1(1) = 3\cdot 1^2 - 2\cdot 1^3 = 3 - 2 = 1$$

$$f_2(1,0) = \frac{2}{3}(1^3-1) - \frac{1}{3}(0^3-0) = 0 - 0 = 0$$

$$f_2(0,1) = \frac{2}{3}(0^3-0) - \frac{1}{3}(1^3-1) = 0 - 0 = 0$$

$$\therefore\ y\Big|_{x_1} = 1\cdot y_1 \quad ✓$$

By symmetry $y\Big|_{x_2} = 1\cdot y_2$ ✓ $(\lambda_2=1,\ \lambda_1=0)$

$$\frac{\partial y}{\partial 12}\Big|_{x_1} = \frac{\partial f_1(\lambda_1)}{\partial \lambda_1}\Big|_{\lambda_1=1} y_1 - \frac{\partial f_1(\lambda_2)}{\partial \lambda_2}\Big|_{\lambda_2=0} y_2$$

$$+\left(\frac{\partial f_2(\lambda_1,\lambda_2)}{\partial \lambda_1} - \frac{\partial f_2(\lambda_1,\lambda_2)}{\partial \lambda_2}\right)\Bigg|_{\substack{\lambda_1=1\\ \lambda_2=0}} \frac{y_1'}{\partial 12} + \left(\frac{\partial f_2(\lambda_2,\lambda_1)}{\partial \lambda_1} - \frac{\partial f_2(\lambda_2,\lambda_1)}{\partial \lambda_2}\right)\Bigg|_{\substack{\lambda_1=1\\ \lambda_2=0}} \frac{y_2'}{\partial 21}$$

$$= \left(6\lambda_1 - 6\lambda_1^2\right)\Big|_{\lambda_1=1} y_1 - \left(6\lambda_2 - 6\lambda_2^2\right)\Big|_{\lambda_2=0} y_2$$

$$+\left[\frac{2}{3}(3\lambda_1^2-1) - -\frac{1}{3}(3\lambda_2^2-1)\right]\Bigg|_{\substack{\lambda_1=1\\ \lambda_2=0}} \frac{y_1'}{\partial 12} + \left[-\frac{1}{3}(3\lambda_1^2-1) - \frac{2}{3}(3\lambda_2^2-1)\right]\Bigg|_{\substack{\lambda_1=1\\ \lambda_2=0}} \frac{y_2'}{\partial 21}$$

$$= 0\cdot y_1 - 0\cdot y_2$$

$$+\left(\frac{2}{3}\cdot 2 - \frac{1}{3}\right)\frac{y_1'}{\partial 12} + \left(-\frac{2}{3} + \frac{2}{3}\right)\frac{y_2'}{\partial 21} \quad (0)$$

$$= \frac{y_1'}{\partial 12} \quad ✓$$

By symmetry $\dfrac{\partial y}{\partial 21}\Big|_{x_2} = \dfrac{y_2'}{\partial 21}$

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$n=2$:

$$y = f_1(\lambda_1)\,y_1 + f_1(\lambda_2)\,y_2 + f_1(\lambda_3)\,y_3$$

$$+ f_2(\lambda_1,\lambda_2,\lambda_3)\,\frac{y_1'}{\partial 12} + f_2(\lambda_2,\lambda_1,\lambda_3)\,\frac{y_2'}{\partial 21}$$

$$+ f_2(\lambda_2,\lambda_3,\lambda_1)\,\frac{y_2'}{\partial 23} + f_2(\lambda_3,\lambda_2,\lambda_1)\,\frac{y_3'}{\partial 32}$$

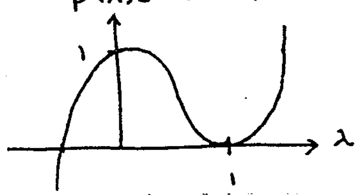
$$+ f_2(\lambda_3,\lambda_1,\lambda_2)\,\frac{y_3'}{\partial 31} + f_2(\lambda_1,\lambda_3,\lambda_2)\,\frac{y_1'}{\partial 13}$$

where $f_1(\lambda) = 3\lambda^2 - 2\lambda^3$

$$f_2(\lambda_1,\lambda_2,\lambda_3) = \left[\tfrac{2}{3}(\lambda_1^3-\lambda_1) - \tfrac{1}{3}(\lambda_2^3-\lambda_2)\right](2\lambda_3^3-3\lambda_3^2+1)$$

Note that $2\lambda_3^3 - 3\lambda_3^2 + 1 = (1-\lambda_3)(1-\lambda_3)(1+2\lambda_3)$

$p(\lambda) = 2\lambda^3 - 3\lambda^2 + 1$



$p(0)=1$
$p(1)=0$
$p'(0)=0$
$p'(1)=0$

The only change from the case for $n=1$ is the addition of $p(\lambda_3)$.

$$\underset{n=2}{f_2(\lambda_1,\lambda_2,\lambda_3)} = \underset{n=1}{f_2(\lambda_1,\lambda_2)}\;p(\lambda_3)$$

When we are on the line from x_1 to x_2 we have $\lambda_3=0$ and $\frac{\partial p(\lambda_3)}{\partial \vec{v}} = 0$, $p(\lambda_3)=1$.

Thus for $\frac{\partial f_2}{\partial 12}$ we find that $p(\lambda_3)$ acts like a constant whose value is one.

$$p(\lambda) = 2\lambda^3 - 3\lambda^2 + 1 = (1-\lambda)^2(1+2\lambda)$$

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Verification of formula:

$$
\begin{aligned}
y\Big|_{x_1,\ \lambda_1=1,\ \lambda_2=0,\ \lambda_3=0} = {} & 1\cdot y_1 + 0\cdot y_2 + 0\cdot y_3 \\
& + \left[\tfrac{2}{3}(1-1) - \tfrac{1}{3}(0-0)\right] p(0)\ \frac{y_1'}{\partial 12} \\
& + \left[\tfrac{2}{3}(0-0) - \tfrac{1}{3}(1-1)\right] p(0)\ \frac{y_2'}{\partial 21} \\
& + \left[\tfrac{2}{3}(0-0) - \tfrac{1}{3}(0-0)\right]\cdot p(1)\ \frac{y_2'}{\partial 23} \\
& + \left[\tfrac{2}{3}(0-0) - \tfrac{1}{3}(0-0)\right] p(1)\ \frac{y_3'}{\partial 32} \\
& + \left[\tfrac{2}{3}(0-0) - \tfrac{1}{3}(1-1)\right] p(0)\ \frac{y_3'}{\partial 31} \\
& + \left[\tfrac{2}{3}(1-1) - \tfrac{1}{3}(0-0)\right] p(0)\ \frac{y_1'}{\partial 13} \\
= {} & y_1 \quad ✓
\end{aligned}
$$

$\vec{v} = x_1$ to x_2

$\partial/\partial\lambda_1 = \cdot(+1)$

$\partial/\partial\lambda_2 = \cdot(-1)$

$\partial/\partial\lambda_3 = 0$

$$
\begin{aligned}
\frac{\partial y}{\partial \vec{v}}\Big|_{x_1,\ \lambda_1=1,\ \lambda_2=0,\ \lambda_3=0} = {} & (6-6)\ y_1 + (-0--0)\ y_2 + 0\ y_3 \\
& + \left[\tfrac{2}{3}(\underbrace{3-1}_{=2}) + \tfrac{1}{3}(\underbrace{3\cdot 0-1}_{=-1})\right] \underbrace{p(0)}_{=1}\ \frac{y_1'}{\partial 12} \\
& + \left[\tfrac{2}{3}(\underbrace{3\cdot 0-1}_{=-1}) - \tfrac{1}{3}(\underbrace{3\cdot 1-1}_{=2})\right] \underbrace{p(0)}_{=1}\ \frac{y_2'}{\partial 21} \\
& + \left[\tfrac{2}{3}(3\cdot 0-1) - \tfrac{1}{3}\ 0\right](+\underbrace{p'(1)}_{=0})\ \frac{y_2'}{\partial 23} \\
& + \left[\tfrac{2}{3}\ 0 + \tfrac{1}{3}(3\cdot 0-1)\right] \underbrace{p'(1)}_{=0}\ \frac{y_3'}{\partial 32} \\
& + \left[\tfrac{2}{3}\ 0 - \tfrac{1}{3}(3\cdot 1-1)\right](-\underbrace{p'(0)}_{=0})\ \frac{y_3'}{\partial 31} \\
& + \left[\tfrac{2}{3}(3\cdot 1-1) - \tfrac{1}{3}\ 0\right](-\underbrace{p'(0)}_{=0})\ \frac{y_1'}{\partial 13} \\
= {} & \frac{y_1'}{\partial 12} \quad ✓
\end{aligned}
$$

Note how the $p(\lambda)$ term helps to eliminate unwanted terms.

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$n \geq 2$: $$y = \sum_{i=1}^{n+1} f_1(\lambda_i)\, y_i$$

$$\sum_{\substack{(i_1,\ldots,i_{n+1}) = \\ \text{all permutations} \\ \text{of } (1,\ldots,n+1)}} f_2(\lambda_{i_1}, \lambda_{i_2})\, p(\lambda_{i_3}) \cdots p(\lambda_{i_{n+1}})\, \frac{y'_{i_1}}{\partial i_1 i_2}$$

where $$f_1(\lambda_i) = 3\lambda_i^2 - 2\lambda_i^3$$

$$f_2(\lambda_{i_1}, \lambda_{i_2}) = \frac{2}{3}(\lambda_{i_1}^3 - \lambda_{i_1}) - \frac{1}{3}(\lambda_{i_2}^3 - \lambda_{i_2})$$

$$p(\lambda) = 2\lambda^3 - 3\lambda^2 + 1 = (1-\lambda)^2(1+2\lambda)$$

For the 1st derivative terms we get a contribution only if none of the p's is differentiated and all the p's have an argument of 0.

Note that each edge of a Δ is a cubic spline, and the spline is the same for all Δ's sharing an edge. Thus, we have continuity along the edges. We may not have smoothness, however, on the edges.