TOOL: The variables Z and χ^2, used in the derivation of the t-distribution are independent.

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

The relevant definitions are as follows:

$n \equiv$ number of data points, X_i, (which are independent and normally distributed)

$\nu \equiv n - 1$

$$\overline{X} \equiv \frac{1}{n}\sum_{i=1}^{n} X_i \equiv \text{sample mean}$$

$$S^2 \equiv \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \equiv \text{sample variance}$$

$$Z \equiv \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \qquad \text{and} \qquad \chi^2 \equiv \nu\frac{S^2}{\sigma^2}$$

PROOF: For convenience, we define the following standard normal random variables:

$$Z_i \equiv \frac{X_i - \mu}{\sigma}$$

We may define Z in terms of the Z_i:

$$Z \equiv \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \sqrt{n}\,\frac{1}{n}\sum_{i=1}^{n}\frac{X_i - \mu}{\sigma} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i$$

We may also define χ^2 in terms of the Z_i:

$$\chi^2 = \frac{n-1}{\sigma^2}\cdot\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma} - \frac{\overline{X} - \mu}{\sigma}\right)^2$$

or

$$\chi^2 = \sum_{i=1}^{n}\left(Z_i - \frac{1}{n}\sum_{j=1}^{n} Z_j\right)^2$$

To prove the independence of Z and χ^2, we show that $E(Z\cdot\chi^2) = E(Z)E(\chi^2)$.

Since $E(Z) = 0$, however, this means we will show that $E(Z\cdot\chi^2) = 0$.

We employ our definitions of Z and χ^2 in terms Z_i's at the outset:

$$E(Z\cdot\chi^2) = E\left(\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} Z_i \cdot \sum_{i=1}^{n}\left(Z_i - \frac{1}{n}\sum_{j=1}^{n} Z_j\right)^2\right)$$

We may rewrite the second summation as follows, (see [1], p. 233):

$$E(Z\cdot\chi^2) = \frac{1}{\sqrt{n}} E\left(\sum_{i=1}^{n} Z_i \cdot \left[\sum_{i=1}^{n} Z_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} Z_i\right)^2\right]\right)$$

Multiplying by n/n and rearranging slightly yields the following expression:

$$E(Z\cdot\chi^2) = \frac{1}{n\sqrt{n}} E\left(n\sum_{i=1}^{n} Z_i^2 \sum_{i=1}^{n} Z_i \cdot - \left(\sum_{i=1}^{n} Z_i\right)^3\right)$$

The Z_i are independent with mean zero, allowing us to eliminate terms that involve products of different Z's. In other words, the only terms that will contribute to the expected value will be those involving a Z_i that is cubed.

$$E(Z\cdot\chi^2) = \frac{1}{n\sqrt{n}} E\left(n\sum_{i=1}^{n} Z_i^3 - \sum_{i=1}^{n} Z_i^3\right) = \frac{1}{n\sqrt{n}} E\left((n-1)\sum_{i=1}^{n} Z_i^3\right)$$

$$E(Z\cdot\chi^2) = \frac{n-1}{n\sqrt{n}} E\left(\sum_{i=1}^{n} Z_i^3\right) = \frac{n-1}{n\sqrt{n}} nE\left(Z_1^3\right) = \frac{n-1}{\sqrt{n}} E\left(Z_1^3\right)$$

Since a standard normal distribution is symmetric around zero, the expected value of each cubed term is zero, however.

$$E(Z\cdot\chi^2) = 0$$

It follows that Z and χ^2 are independent.

REF: [1] Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, and Keying Ye, *Probability and Statistics for Engineers and Scientists,* 8th Ed., Upper Saddle River, NJ: Prentice Hall, 2007.