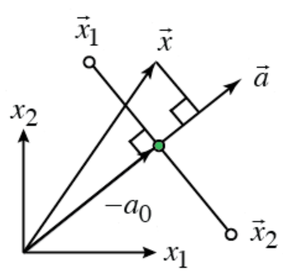
TOOL: The following procedure defines decision boundaries that may be used to determine whether a point lies in a triangle. The rationale is that a given point, $\vec{x}$, is on the inside of the triangle if and only if $\vec{x}$ is on the correct side of each edge of the triangle. A vector, $\vec{a}$, is found for each side of the triangle such that the dot product of $\vec{a}$ and $\vec{x}$ is greater than a know constant if and only if $\vec{x}$ is on the correct side of a particular edge of the triangle. If dot products are large enough for all sides of the triangle, then $\vec{x}$ is inside the triangle. The mathematics of defining the vector $\vec{a}$ is equivalent to creating a decision boundary for a perceptron where the decision boundary corresponds to an edge of the triangle. Thus, $\vec{a}$ is referred to as a decision-boundary vector. The mathematical details of the procedure for finding $\vec{a}$ follow. A 2-dimensional case is described, but the process generalizes to N dimensions in an obvious way.

i) Given two points, $\vec{x}_1$ and $\vec{x}_2$, (or N points in N dimensions), defining an edge of a triangle, the decision boundary vector, $\vec{a}$, is perpendicular to the line segment from $\vec{x}_1$ to $\vec{x}_2$. (In N-dimensions, the decision boundary vector, $\vec{a}$, is perpendicular to a hyper-plane containing vertices of one side, meaning all but one vertex, of the N-dimensional tetrahedron.) It follows that which side of the edge a point $\vec{x}$ lies on may be found by computing the dot product of $\vec{x}$ with $\vec{a}$ and comparing it to an appropriate constant value, $-a_0$. In particular, the dot product of $\vec{x}_1$ and $\vec{x}_2$ with $\vec{a}$ should give the same value, $-a_0$. (These calculations correspond to projecting $\vec{x}$, $\vec{x}_1$, and $\vec{x}_2$ onto $\vec{a}$.)



$$\begin{bmatrix} \vec{x}_1^T \\ \vec{x}_2^T \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -a_0 \\ -a_0 \end{bmatrix}$$

ii) These calculations may be rearranged to create a matrix equation. The equations, however is underdetermined.

$$\begin{bmatrix} 1 & \vec{x}_1^T \\ 1 & \vec{x}_2^T \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

iii) To create a solvable system of equations, a third row is needed in the matrix. In addition, the right side is zero. This means that the augmented $\vec{a}$, which is denoted as $\vec{a}_+$, is unique only to within a scaling constant. Consequently, a third-row equation may be created by adding a third vector whose dot product with $\vec{a}_+$ equals an arbitrary nonzero value. This eliminates both the problem of a zero right side and a non-unique solution. The vector forming the third row of the matrix, however, must be linearly independent of the other two vectors. To find such a vector, a cross product may be used. The cross product, which produces a vector, $\vec{n}$, that is perpendicular (normal) to the vectors in the cross product is computed as the determinant of a matrix [1]:

$$\vec{n} = (1, \vec{x}_1) \times (1, \vec{x}_2) \times \ldots \times (1, \vec{x}_{N-1}) = \begin{vmatrix} \vec{e}_0 & \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_N \\ 1 & x_{11} & x_{12} & \cdots & x_{1N} \\ 1 & x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N-1,1} & x_{N-1,2} & \cdots & x_{N-1,N} \end{vmatrix}$$

where $\vec{e}_i \equiv$ unit vector in direction of ith axis

iv) The cross product is added to the matrix to create a solvable system of equations that yields a decision-boundary vector, $\vec{a}_+ = (a_0, \vec{a})^T$.

$$\begin{bmatrix} 1 & \vec{x}_1^T \\ 1 & \vec{x}_2^T \\ & \vec{n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{a}_+ = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & \vec{x}_1^T \\ 1 & \vec{x}_2^T \\ & \vec{n} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

v) The final step of the procedure is to determine whether $\vec{a}_+$ gives a positive result for points on the inside of the triangle under consideration. This result is determined by computing the dot product $s = \vec{a}_+^T \circ (1, \vec{x}_{N+1})$ where $\vec{x}_{N+1}$ is the unused vertex, (which must lie on the side of the edge toward the inside of the triangle). If $s < 0$, then $\vec{a}_+$ is replaced with $-\vec{a}_+$. The calculation of $s = \vec{a}_+^T \circ (1, \vec{x})$ now indicates $\vec{x}$ is on the inside of the edge when $s > 0$.

REF: [1] Helmut K. Fishbeck and Kurt H. Fishbeck, *Formula,s Facts and Constants,* 2nd Ed., Berlin, GDR: Springer-Verlag, 1987.