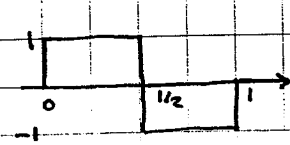
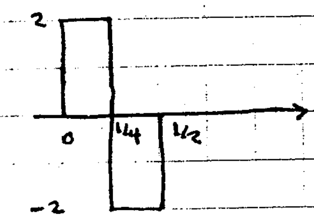
Neil E Co

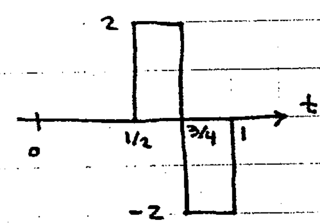
ex: Haar wavelets



$= \psi_{00}(t)$



$\psi_{-1,0}(t)$



The wavelets are orthogonal because they either do not overlap at all, or one is constant over the interval where the second one varies.

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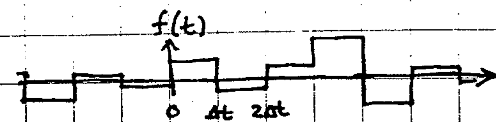
ex: Haar wavelets form a complete basis for step functions on the real line with regularly spaced steps, (and having finite energy).

def: step function $\equiv$ $f(t)$ is constant over intervals of positive measure. The intervals follow immediately after one another and cover the real line.

ex:



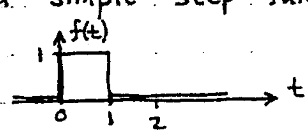
For this example, the intervals are regularly spaced every Δt. In general, this need not be the case, but the intervals must have positive length.

pf: (that Haar wavelets are complete)

We use multiresolution analysis to show how we can drive the energy of the error waveform (for approximating a step function with wavelets) to zero.

We wish to show $\displaystyle\int_{-\infty}^{\infty}\left[f(t) - \sum_m \sum_n w_{m,n}\, \Psi_{m,n}(t)\right]^2 dt.$

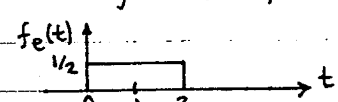
Consider a simple step function $f(t)$:



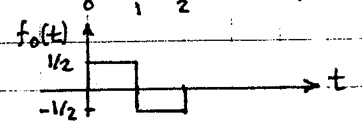
Write $f(t)$ as the sum of:

$f_e(t)$ 1) The average value (even part) of $f(t)$ on interval $[0,2]$, and

$f_o(t)$ 2) $f(t)$ − average (odd part) of $f(t)$ on interval $[0,2]$



$f(t) = f_e(t) + f_o(t)$



We observe that $f_o(t)$, the odd part is a Haar wavelet multiplied by a scaling factor. As we proceed, the odd part is always a Haar wavelet.

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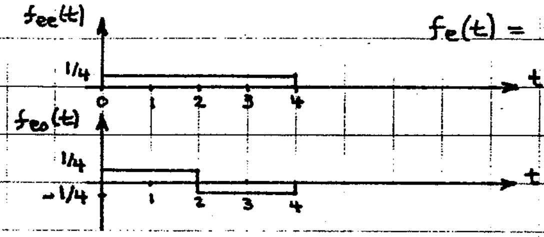
pf: (that Haar wavelets are complete) (cont)

Now are left with $f_e(t)$ to approximate. But $f_e(t)$ is another step function over a longer time interval.

So now we repeat the above process of taking an average and difference, but we use an interval twice as long, namely $[0,4]$:

$$f_e(t) = f_{ee}(t) + f_{eo}(t)$$



$f_{eo}(t)$ is again a Haar wavelet, and we are left with the problem of approximating $f_{ee}(t)$.

Clearly, we can continue this process on wider and wider intervals.

We observe that the area of $\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{\infty} f_e(t)\,dt = \int_{-\infty}^{\infty} f_{ee}(t)\,dt = \ldots = 1.$

Thus, the area under the residual error function is remaining constant.

But, the energy in the error function decreases by a factor 2 at each step.

$$\int_{-\infty}^{\infty} f^2(t)\,dt = 1 \qquad \int_{-\infty}^{\infty} f_e^2(t)\,dt = \frac{1}{2} \qquad \int_{-\infty}^{\infty} f_{ee}^2(t)\,dt = \frac{1}{4}$$

Thus, the energy in the error will go to zero, and we can approximate a step.

To approximate a function with more than one step, we simply approximate each step and sum the wavelet series for each step. (Our procedure, above, is linear: $[f_1(t)+f_2(t)]_e = f_{1e}(t) + f_{2e}(t)$

$[f_1(t)+f_2(t)]_o = f_{1o}(t) + f_{2o}(t)$)

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tool: We can approximate any real-valued continuous function on the real line with a step function. In other words, we can find a sequence of step functions that converges uniformly to the given continuoas function.

It follows that Haar wavelets are a complete basis for finite-energy, real-valued functions on the real line.