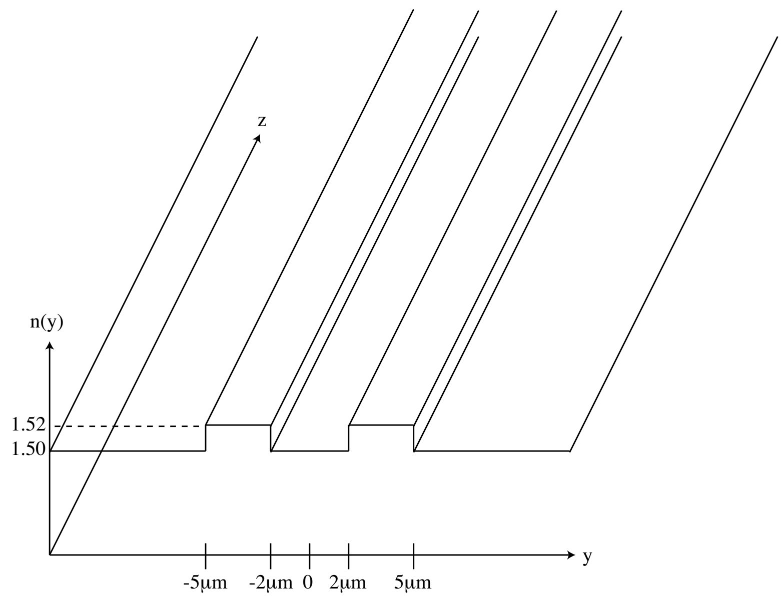
ECE 6440 - Photonic Microsystems
Midterm Exam II Spring 2011

Choose two of the following three problems (somehow indicate which two problems you want graded).

Problem 1 (20 points)

Consider the following directional coupler:



The effective indices from the refractive index distribution along the x-direction are listed for x-polarized light.

a) Solve for the fundamental eigenmode of an isolated waveguide (i.e. solve for β and the mode profile) at wavelength $\lambda = 1.55\ \mu$m.

The waveguide is symmetric. We first calculate the V number

$$V = \frac{2\pi h}{\lambda_f}\sqrt{n_f^2 - n_s^2} = \frac{2\pi \times 3\ \mu\text{m}}{1.55\ \mu\text{m}}\sqrt{1.52^2 - 1.5^2} = 2.99,$$

so there is only one mode. From the waveguide dispersion chart, we obtain $b = 0.62650$, or $n_{eff} = 1.51256$.

For the symmetric TE mode of a symmetric waveguide, we have

$$\mathcal{E}_a(y) = \begin{cases} C\ e^{\gamma(y+5\mu\text{m})} & y < -5\mu\text{m} \\ C\ \frac{\cos[\kappa(y+3.5\mu\text{m})]}{\cos(\kappa h/2)} & -5\mu\text{m} \le y \le -2\mu\text{m} \\ C\ e^{-\gamma(y+2\mu\text{m})} & y > -2\mu\text{m} \end{cases}$$

From β, we can calculate γ and κ:

$$\kappa = \sqrt{k_f^2 - \beta^2} = \frac{2\pi}{\lambda}\sqrt{1.52^2 - 1.51256^2} = 0.6089\ \mu\text{m}^{-1}$$

$$\gamma = \sqrt{\beta^2 - k_c^2} = \frac{2\pi}{\lambda}\sqrt{1.51256^2 - 1.5^2} = 0.7885\ \mu\text{m}^{-1}$$

Now, plugging these values in, we get a field profile:

$$\mathcal{E}_a(y) = \begin{cases} C\ e^{0.7885\mu\text{m}^{-1}(y+5\mu\text{m})} & y < -5\mu\text{m} \\ 1.6364\ C\ \cos\left[0.6089\mu\text{m}^{-1}\left(y + 3.5\mu\text{m}\right)\right] & -5\mu\text{m} \leq y \leq -2\mu\text{m} \\ C\ e^{-0.7885\mu\text{m}^{-1}(y+2\mu\text{m})} & y > -2\mu\text{m} \end{cases}$$

b) Normalize the amplitude of the mode.

In order to normalize the mode, we need to find the value of C for which

$$\int \mathcal{E}_a(y)\mathcal{E}_a(y)dy = \frac{2\omega\mu}{\beta} = \frac{2\mu c}{n_{\text{eff}}} = \frac{2\eta_o}{n_{\text{eff}}}.$$

Plugging in the mode profile, we separate into three integrals

$$\int_{-\infty}^{-5\mu\text{m}} e^{1.577\mu\text{m}^{-1}(y+5\mu\text{m})}dy + 2.6778\int_{-5\mu\text{m}}^{-3.5\mu\text{m}} \cos^2\left[0.6089\mu\text{m}^{-1}\left(y+3.5\mu\text{m}\right)\right]dy = \frac{\eta_o}{C^2 n_{\text{eff}}}$$

Note that a factor of 2 has been taken out due to the symmetry of the problem. Integrating, we get

$$\frac{e^{1.577\mu\text{m}^{-1}(y+5\mu\text{m})}}{1.577\mu\text{m}^{-1}}\Big|_{-\infty}^{-5\mu\text{m}} + \frac{2.6778}{2}\int_{-5\mu\text{m}}^{-3.5\mu\text{m}} \left\{1 + \cos\left[2\times 0.6089\mu\text{m}^{-1}\left(y+3.5\mu\text{m}\right)\right]\right\}dy = \frac{\eta_o}{C^2 n_{\text{eff}}}$$

$$\frac{1}{1.577\mu\text{m}^{-1}}\left[1-0\right] + 1.3389\left\{y + \frac{1}{1.2178\mu\text{m}^{-1}}\sin\left[1.2178\mu\text{m}^{-1}\left(y+3.5\mu\text{m}\right)\right]\right\}\Big|_{-5\mu\text{m}}^{-3.5\mu\text{m}} = \frac{\eta_o}{C^2 n_{\text{eff}}}$$

$$0.6341\mu\text{m} + 1.3389\left(1.5\mu\text{m} + \frac{1}{1.2178\mu\text{m}^{-1}}\left\{0 - \sin\left[1.2178\mu\text{m}^{-1}\left(-1.5\mu\text{m}\right)\right]\right\}\right) = \frac{\eta_o}{C^2 n_{\text{eff}}}$$

$$0.6341\mu\text{m} + 1.3389\left(1.5\mu\text{m} + 0.7944\mu\text{m}\right) = \frac{\eta_o}{C^2 n_{\text{eff}}}$$

$$C^2 = \frac{\eta_o}{(3.7061\mu\text{m})n_{\text{eff}}} = \frac{377\ \Omega}{(3.7061\mu\text{m})(1.5126)} = 67.25\ \Omega/\mu\text{m}$$

Because this problem is degenerate (i.e. the mode of the second waveguide is identical), we now have the normalized forms for $\mathcal{E}_a$ and $\mathcal{E}_b$.

c) Solve for the coupling coefficient ($\mathcal{K}$) between the two waveguides.

The perturbation in this problem is the presence of the second waveguide. The perturbation term is

$$P_{pert} = \frac{\mathcal{E}_a(y)\epsilon_o}{2}\left(n_f^2 - n_s^2\right)e^{-j(\beta z - \omega t)} + \text{c.c.}$$

defined over 2 μm $\leq y \leq$ 5 μm.

The coupling coefficient is defined

$$\begin{aligned}
\mathcal{K} &= \frac{\epsilon_o\omega}{4}\int_{2\mu\mathrm{m}}^{5\mu\mathrm{m}}\left(n_f^2-n_s^2\right)\mathcal{E}_a(y)\mathcal{E}_b(y)dy \\
&= 0.09884\frac{\epsilon_o\omega}{4}C^2\int_{2\mu\mathrm{m}}^{5\mu\mathrm{m}} e^{-0.7885\mu\mathrm{m}^{-1}(y+2\mu\mathrm{m})}\cos\left[0.6089\mu\mathrm{m}^{-1}\left(y-3.5\mu\mathrm{m}\right)\right]dy \\
&= 0.01789\mu\mathrm{m}^{-2}\int_{2\mu\mathrm{m}}^{5\mu\mathrm{m}} e^{-0.7885\mu\mathrm{m}^{-1}(y+2\mu\mathrm{m})}\cos\left[0.6089\mu\mathrm{m}^{-1}\left(y-3.5\mu\mathrm{m}\right)\right]dy
\end{aligned}$$

Now, we must use the integration formula

$$\int\cos(ay)e^{by}dy=\frac{e^{by}}{a^2+b^2}\left[a\sin(ay)+b\cos(ay)\right],$$

where $a=0.6089$ and $b=-0.7885$. With this, we can evaluate $\mathcal{K}$

$$\begin{aligned}
\mathcal{K} &= 0.01789\mu\mathrm{m}^{-2}\left\{\frac{e^{-0.7885\mu\mathrm{m}^{-1}(y+2\mu\mathrm{m})}}{0.3708\mu\mathrm{m}^{-2}+0.6217\mu\mathrm{m}^{-2}}\times\right. \\
&\quad\left.\left[0.6089\cos\left[0.6089\mu\mathrm{m}^{-1}\left(y-3.5\mu\mathrm{m}\right)\right]-0.7885\sin\left[0.6089\mu\mathrm{m}^{-1}\left(y-3.5\mu\mathrm{m}\right)\right]\right]\ \mu\mathrm{m}^{-1}\right\}\Big|_{2\mu\mathrm{m}}^{5\mu\mathrm{m}} \\
&= 0.01789e^{-0.7885\mu\mathrm{m}^{-1}(y+2\mu\mathrm{m})}\times \\
&\quad\left\{0.6089\cos\left[0.6089\mu\mathrm{m}^{-1}\left(y-3.5\mu\mathrm{m}\right)\right]-0.7885\sin\left[0.6089\mu\mathrm{m}^{-1}\left(y-3.5\mu\mathrm{m}\right)\right]\right\}\ \mu\mathrm{m}^{-1}\ \Big|_{2\mu\mathrm{m}}^{5\mu\mathrm{m}} \\
&= 0.01789e^{-0.7885\mu\mathrm{m}^{-1}(7\mu\mathrm{m})}\times \\
&\quad\left\{0.6089\cos\left[0.6089\mu\mathrm{m}^{-1}\left(1.5\mu\mathrm{m}\right)\right]-0.7885\sin\left[0.6089\mu\mathrm{m}^{-1}\left(1.5\mu\mathrm{m}\right)\right]\right\}\ \mu\mathrm{m}^{-1} \\
&\quad -0.01789e^{-0.7885\mu\mathrm{m}^{-1}(4\mu\mathrm{m})}\times \\
&\quad\left\{0.6089\cos\left[0.6089\mu\mathrm{m}^{-1}\left(1.5\mu\mathrm{m}\right)\right]+0.7885\sin\left[0.6089\mu\mathrm{m}^{-1}\left(1.5\mu\mathrm{m}\right)\right]\right\}\ \mu\mathrm{m}^{-1} \\
&= 0.00007223\times\{0.3721-0.6241\}\ \mu\mathrm{m}^{-1}-0.0007695\times\{0.3721+0.6241\}\ \mu\mathrm{m}^{-1} \\
&= (-0.0000182-0.0007665)\ \mu\mathrm{m}^{-1}=-0.0007852\ \mu\mathrm{m}^{-1}.
\end{aligned}$$

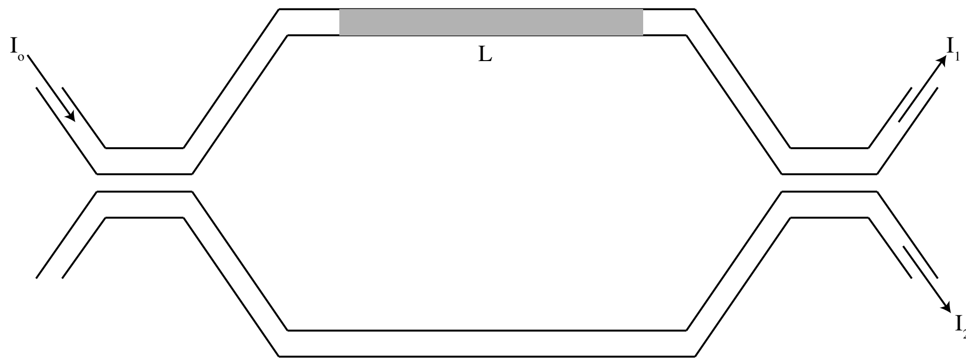
d) What is the physical length over which complete coupling occurs from one waveguide to the other?

The coupling length for complete transfer is

$$Z_o=\frac{\pi}{2\mathcal{K}}=2.0\text{ mm}.$$

Problem 2 (20 points)

You are designing a Mach-Zehnder amplitude moulator using silicon, as shown in the following figure: The free-space wavelength is $\lambda = 1.55\ \mu$m.



a) For a maximum carrier density of $N = 1 \times 10^{17}$ cm^{-3}, what is the modulation length L required to obtain a π phase shift in the upper arm?

For Si at a wavelength 1.55 μm, the change in refractive index and absorption coefficient

$$\begin{aligned}\Delta n &= -8.8 \times 10^{-22} N_e - 8.5 \times 10^{-18} N_h^{0.8} \\ \Delta\alpha &= 8.5 \times 10^{-18} N_e + 6 \times 10^{-18} N_h.\end{aligned}$$

Plugging $N_e = N_h = 10^{17}$ cm^3,

$$\begin{aligned}\Delta n &= -8.8 \times 10^{-22} \times 10^{17} - 8.5 \times 10^{-18} \times \left(10^{17}\right)^{0.8} = -0.00043 \\ \Delta\alpha &= 8.5 \times 10^{-18} \times 10^{17} + 6 \times 10^{-18} \times 10^{17} = 1.45\ \text{cm}^{-1}.\end{aligned}$$

For a π phase shift, we need

$$\Delta\phi \equiv \pi = \Delta\beta L \approx \frac{2\pi}{\lambda}\Delta n L = 0.0027\frac{L}{\lambda},$$

or

$$L = \frac{\pi\lambda}{0.0027} = 1173\ \lambda = 1802\ \mu\text{m}.$$

b) Taking into account both change in refractive index and absorption coefficient with carrier density, calculate the intensity of each output of the interferometer given an input intensity of I_o for both the "on" and "off" states. Assume the couplers are 50/50.

We can write the input-output relationship of the Mach-Zehnder as

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -j \\ -j & 1 \end{bmatrix}\begin{bmatrix} ae^{-j\Delta\beta L} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -j \\ -j & 1 \end{bmatrix}\begin{pmatrix} E_0 \\ 0 \end{pmatrix},$$

where $a = e^{-\Delta\alpha L/2}$.

Multiplying out the matrices

$$\begin{aligned}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} &= \frac{1}{2}\begin{bmatrix} 1 & -j \\ -j & 1 \end{bmatrix}\begin{bmatrix} ae^{-j\Delta\beta L} & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} E_0 \\ -jE_0 \end{pmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -j \\ -j & 1 \end{bmatrix}\begin{pmatrix} ae^{-j\Delta\beta L}E_0 \\ -jE_0 \end{pmatrix} \\ &= \frac{E_0}{2}\begin{pmatrix} ae^{-j\Delta\beta L} - 1 \\ -jae^{-j\Delta\beta L} - j \end{pmatrix}\end{aligned}$$

Now, writing the output intensities

$$I_1 = \left|\frac{E_1}{E_0}\right|^2 = \frac{1}{4}\left|ae^{-j\Delta\beta L} - 1\right|^2 = \frac{1}{4}\left[a^2 - 2a\cos\left(\Delta\beta L\right) + 1\right]$$

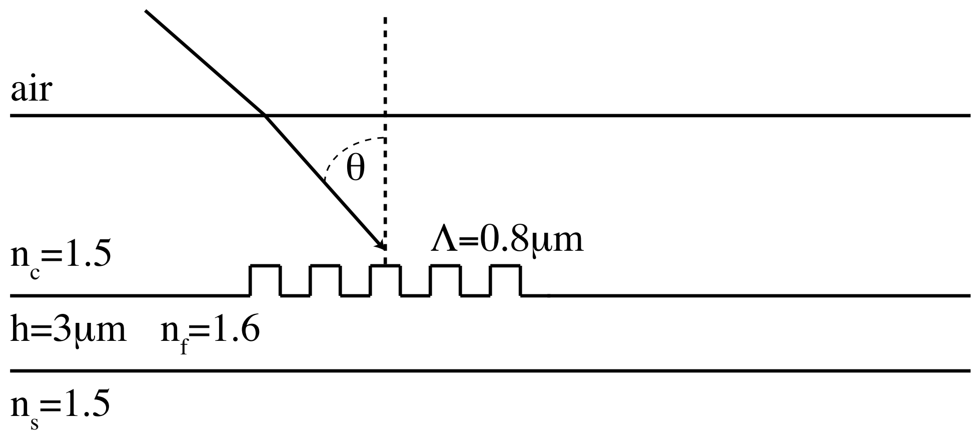
$$I_2 = \left|\frac{E_1}{E_0}\right|^2 = \frac{1}{4}\left|ae^{-j\Delta\beta L} + 1\right|^2 = \frac{1}{4}\left[a^2 + 2a\cos\left(\Delta\beta L\right) + 1\right]$$

In the "off" state, $a = 1$ and $\Delta\beta L = 0$, so that $I_1 = 0$ and $I_2 = 1$.

In the "on" state, $\Delta\beta L = \pi$ and $\Delta\alpha L = 0.261$ so that $a = 0.878$. In this state, $I_1 = 0.882$ and $I_2 = 0.004$, where light intensity is absorbed in the upper arm.

Problem 3 (20 points)

For grating coupling into a slab waveguide using a free-space wavelength $\lambda = 1.55\ \mu$m,



a) What are the effective indices of the TE modes?

The waveguide is symmetric. We first calculate the V number

$$V = \frac{2\pi h}{\lambda_f}\sqrt{n_f^2 - n_s^2} = \frac{2\pi \times 3\ \mu\text{m}}{1.55\ \mu\text{m}}\sqrt{1.6^2 - 1.5^2} = 6.77,$$

so there are three modes. From the waveguide dispersion chart, we obtain $b = 0.87301, 0.51106, 0.03408$ and $n_{eff} = 1.58765, 1.55191, 1.50352$.

b) What are the internal and external incident angle for coupling into these modes via the grating?

First, for the internal coupling angles (angles with respect to normal in the cladding layer), we must satisfy the first-order grating coupling condition:

$$\frac{2\pi n_c}{\lambda}\sin\theta + \frac{2\pi}{\Lambda} = \frac{2\pi n_{eff}}{\lambda}$$

or, simplified,

$$n_c \sin\theta = n_{eff} - \frac{\lambda}{\Lambda}.$$

Using the lowest-order mode as an example, we get an internal angle

$$1.5\sin\theta = 1.58765 - \frac{1.55}{0.8} = -0.34985,$$

giving an angle $\theta = -13.5°$. For the other waveguide modes, we get -14.9° and -16.8°.

To calculate the external angles, we use Snell's Law

$$\sin\theta_{\text{ext}} = n_c \sin\theta,$$

with the external angles: -20.5, -22.7, and -25.7°.