2/25/09

Numerical Methods

for wave propagation in a waveguide, we have two major choices for starting:

Maxwell's equations

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\vec{D} = \epsilon(x,y,z)\vec{E}$$

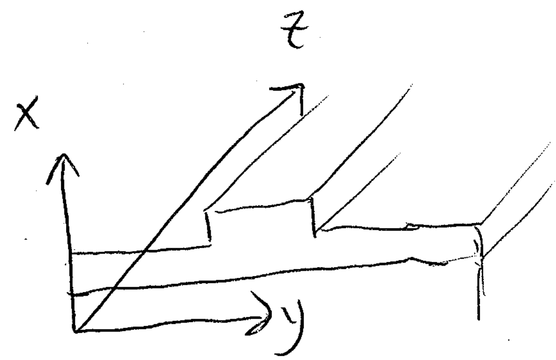
Wave equation

$$\nabla^2 \vec{E} - \mu_0 \epsilon(x,y,z) \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$\downarrow$ scalar, harmonic

$$\nabla^2 A + \frac{\omega^2}{c^2} n^2(x,y,z) A = 0$$

$$E = \frac{1}{2} A e^{-j\omega t} + cc$$



Finite difference time domain method (FDTD)

works directly w/ Maxwell's equations

1) retains carrier, envelope, group velocity, etc.
2) must sample $\Delta x, \Delta y, \Delta z \ll \lambda$, $\Delta t \ll 2\pi/\omega$ $\left(\Delta t \approx \frac{\Delta x}{2c}\right)$ stability criterion
3) includes evanescent waves, backward waves

consider 2-D TE case:

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu_0}\left(-\frac{\partial E_z}{\partial y}\right) \qquad \frac{\partial H_y}{\partial t} = \frac{1}{\mu_0}\frac{\partial E_z}{\partial x}$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon(x,y)}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

now, we finite difference:

$$\frac{\partial H_x}{\partial t} = \frac{H_x\Big|^{n+1}_{i,j+1/2} - H_x\Big|^{n}_{i,j+1/2}}{\Delta t}$$

$$\frac{\partial E_z}{\partial y} = \frac{E_z\Big|^{n+1/2}_{i,j+1} - E_z\Big|^{n+1/2}_{i,j}}{\Delta y}$$

$$\Rightarrow \quad H_x\Big|^{n+1}_{i,j+1/2} = \frac{\Delta t}{\mu_0 \Delta y}\left(E_z\Big|^{n+1/2}_{i,j} - E_z\Big|^{n+1/2}_{i,j+1}\right) + H_x\Big|^{n}_{i,j+1/2}$$

notice symmetries, 2nd order accuracy in $\Delta t, \Delta y$

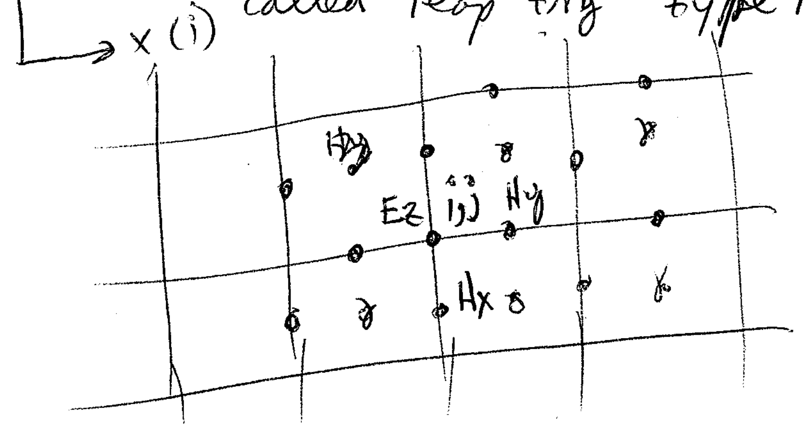
$$H_y\Big|^{n+1}_{i+1/2,j} = \frac{\Delta t}{\mu_0 \Delta x}\left(E_z\Big|^{n+1/2}_{i+1,j} - E_z\Big|^{n+1/2}_{i,j}\right) + H_y\Big|^{n}_{i+\frac{1}{2},j}$$

$$E_z\Big|^{n+1/2}_{i,j} = \frac{\Delta t}{\varepsilon(x,y)}\left(\frac{H_y\Big|^{n}_{i+\frac{1}{2},j} - H_y\Big|^{n}_{i-\frac{1}{2},j}}{\Delta x} + \frac{H_x\Big|^{n}_{i,j-\frac{1}{2}} - H_x\Big|^{n}_{i,j+\frac{1}{2}}}{\Delta y}\right) + E_z\Big|^{n-1/2}_{i,j}$$

notice also the staggered grid, E, H are offset in space by $\frac{\Delta x}{2}, \frac{\Delta y}{2}$ and in time by $\frac{\Delta t}{2}$

called "leap frog" type method

Yee cell



Finite difference method

$$\frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial x^2} + \frac{\omega^2}{c^2} n^2(x) A = 0$$

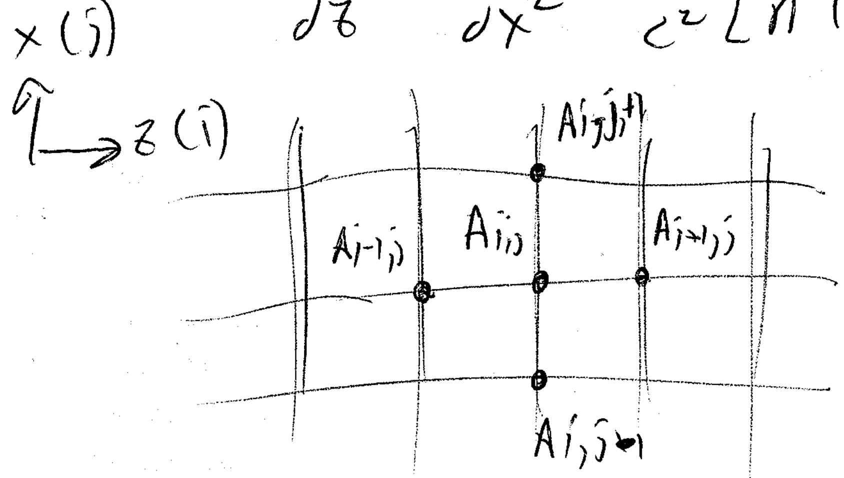
assume index variation only along x, propagate along z

$$A \rightarrow \tilde{A} e^{jkz} \qquad k_0 = \frac{\omega}{c} n_0 \qquad n_0 \text{ is average}$$

$$\frac{\partial^2 A}{\partial z^2} = \frac{\partial^2 \tilde{A}}{\partial z^2} e^{jkz} + 2jk_0 \frac{\partial \tilde{A}}{\partial z} e^{jkz} - k_0^2 \tilde{A} e^{jkz}$$

assume $\frac{\partial^2 \tilde{A}}{\partial z^2} \ll 2k_0 \frac{\partial \tilde{A}}{\partial z}$ ~~SVE~~ SVAA

$$2jk_0 \frac{\partial \tilde{A}}{\partial z} + \frac{\partial^2 \tilde{A}}{\partial x^2} + \frac{\omega^2}{c^2}\left[n^2(x) - n_0^2\right]\tilde{A} = 0$$



center about ~~solve for~~ $A_{i,j}^{+1/2}$

$$\frac{\partial A}{\partial z} \rightarrow \frac{A_{i+1,j} - A_{i-1,j}}{2\Delta z}$$

$$\frac{\partial^2 A}{\partial x^2} \rightarrow \frac{1}{2}\left(\frac{A_{i,j+1} - 2A_{i,j} + A_{i,j-1}}{\Delta x^2} + \frac{A_{i+1,j+1} - 2A_{i+1,j} + A_{i+1,j-1}}{\Delta x^2}\right)$$

$$\frac{jk_0}{\Delta z}\left(A_{i+1,j} - A_{i-1,j}\right) + \frac{1}{2\Delta x^2}\left(A_{i,j+1} - 2A_{i,j} + A_{i,j-1}\right) + \frac{1}{2\Delta x^2}(\ )$$

$$+ \frac{\omega^2}{c^2}\left(n^2(x) - n_0^2\right)\left(\frac{A_{i+1,j} + A_{i,j}}{2}\right) = 0$$

solve for $\cancel{A_{i+1,j}}$ in terms of others

$$\frac{jk_0}{\Delta z}\left(A_{i+1,j}-A_{i,j}\right)+\frac{\cancel{\omega}}{4\cancel{\omega}\Delta x^2}\Big[\left(A_{i,j+1}-2A_{i,j}+A_{i,j-1}\right)+\left(A_{i+1,j+1}-2A_{i+1,j}+A_{i+1,j-1}\right)\Big]+\frac{\omega^2}{4c^2}\left[n^2(x)-n_0^2\right]\times\left(A_{i+1,j}+A_{i,j}\right)=0$$

pull A_{i+1} terms to left, these are unknowns

$$\frac{jk_0}{\Delta z}A_{i+1,j}+\frac{\cancel{\omega}}{4\cancel{\omega}\Delta x^2}\left(A_{i+1,j+1}-2A_{i+1,j}+A_{i+1,j-1}\right)+\frac{\omega^2}{4c^2}\left[n^2(x)-n_0^2\right]A_{i+1,j}=\frac{jk_0}{\Delta z}A_{i,j}-\frac{\cancel{\omega}^2}{4\cancel{\omega}\Delta x^2}\left(A_{i,j+1}-2A_{i,j}+A_{i,j-1}\right)-\frac{\omega^2}{4c^2}\left[n^2(x)-n_0^2\right]\times A_{i,j}$$

$$a\,A_{i+1,j+1}+b_1\,A_{i+1,j}+a\,A_{i+1,j-1}=-a\,A_{i,j+1}-b_2\,A_{i,j}-a\,A_{i,j-1}$$

$$a=\frac{\cancel{\omega}}{4\cancel{\omega}\Delta x^2}\qquad b_1=-2a+\frac{jk_0}{\Delta z}+\frac{\omega^2}{4c^2}\left[n_j^2-n_0^2\right]$$

$$b_2=-2a-\frac{jk_0}{\Delta z}+\frac{\omega^2}{4c^2}\left[n_j^2-n_0^2\right]$$

tri-diagonal matrix

$$\begin{pmatrix} +b_1 & a & & \\ a & +b_1 & a & \\ & a & +b_1 & a \\ & & & \ddots \end{pmatrix} \begin{pmatrix} A_{i+1,0} \\ A_{i+1,1} \\ A_{i+1,2} \\ \vdots \\ \vdots \\ A_{i+1,N} \end{pmatrix} = \begin{pmatrix} rhs_0 \\ rhs_1 \\ rhs_2 \\ \vdots \\ \vdots \\ rhs_N \end{pmatrix}$$

$$rhs0 = -b_2 A_{i,0} - a A_{i,1}$$
$$rhs1 = -a A_{i,0} - b_2 A_{i,1} - a A_{i,2}$$
$$\vdots$$

we are just solving for next z position $z + \Delta z$
can only do forward propagation.
no requirement that $\Delta z \ll ?$, however
tridiagonal matrix can be solved in $O(N)$ steps
if we had x, y variation, need to use ADI method.

The Equations We Actually Use

$$\left(\tau\frac{\partial}{\partial t}+1\right)\tilde{E}^{SC}=\frac{\partial^2\tilde{E}^{SC}}{\partial\tilde{x}^2}-\frac{\partial I/\partial\tilde{x}}{(I+I_d)}$$

- slow variation of space-charge field: $\psi \approx 1$
- what we can't deal with: $\tau_2 \rightarrow 0$
- large donar density: $N_D \gg N_A$

$$2ik\frac{\partial A}{\partial z}+\frac{\partial^2 A}{\partial x^2}+4k\gamma_o\tilde{E}^{SC}A=0$$

- $E(x,z,t)=A(x,z)e^{i(kz-\omega t)}$
- paraxial approximation
- coupling constant: $\gamma_o = 2\omega n^3 r_{eff}/c$
- r_{eff}: effective electro-optic coefficient

Crank-Nicholson Finite-Difference

$$\left(\tau\frac{\partial}{\partial t}+1\right)\tilde{E}=\frac{1}{k_D^2}\cdot\frac{\partial^2\tilde{E}}{\partial x^2}-\frac{1}{k_D}\cdot\frac{\partial I/\partial x}{(I+I_d)}$$

- $E_i^n \equiv E(i\Delta x, n\Delta t)$
- center about $E_i^{n+1/2}$

$$E \approx \frac{1}{2}\left(E_i^{n+1}+E_i^n\right)$$

$$\frac{\partial E}{\partial t}\approx\frac{E_i^{n+1}-E_i^n}{\Delta t}$$

$$\frac{\partial I}{\partial x}\approx\frac{I_{i+1}^n-I_{i-1}^n}{2\Delta x}$$

$$\frac{\partial^2 E}{\partial x^2}\approx\frac{1}{2}\left(\frac{E_{i+1}^{n+1}-2E_i^{n+1}+E_{i-1}^{n+1}}{\Delta x^2}+\frac{E_{i+1}^n-2E_i^n+E_{i-1}^n}{\Delta x^2}\right)$$

Differenced Equation

$$\frac{1}{2k_D^2}\left(\frac{E_{i+1}^{n+1}-2E_i^{n+1}+E_{i-1}^{n+1}}{\Delta x^2}+\frac{E_{i+1}^{n}-2E_i^{n}+E_{i-1}^{n}}{\Delta x^2}\right)$$

$$-\frac{1}{2k_D\Delta x}\cdot\frac{I_{i+1}^n-I_{i-1}^n}{I_i^n+I_d}=\frac{\tau}{\Delta t}\left(E_i^{n+1}-E_i^n\right)+\frac{1}{2}\left(E_i^{n+1}+E_i^n\right)$$

$$aE_{i+1}^{n+1}-b_1E_i^{n+1}+aE_{i-1}^{n+1}=$$

$$-aE_{i+1}^{n}+b_2E_i^{n}-aE_{i-1}^{n}-c\frac{I_{i+1}^n-I_{i-1}^n}{I_i^n+I_d}$$

- $a \equiv 1/2k_D^2\Delta x^2$
- $b_1 \equiv 2a+\tau/\Delta t+1/2$
- $b_2 \equiv 2a-\tau/\Delta t+1/2$
- $c \equiv 1/2k_D\Delta x$

Tri-Diagonal Matrix

$$\begin{pmatrix} -b_1 & a & & & \\ a & -b_1 & a & & \\ & a & -b_1 & a & \\ & & \cdots & & \\ & & & a & -b_1 \end{pmatrix} \cdot \begin{pmatrix} E_0^{n+1} \\ E_1^{n+1} \\ E_i^{n+1} \\ \cdots \\ E_{max-1}^{n+1} \end{pmatrix} = \begin{pmatrix} rhs_0^n \\ rhs_1^n \\ rhs_i^n \\ \cdots \\ rhs_{max-1}^n \end{pmatrix}$$

- $rhs_0^n = -aE_1^n + b_2 E_0^n - c\, I_1^n/(I_0^n + I_d)$
- $rhs_i^n = -aE_{i+1}^n + b_2 E_i^n - aE_{i-1}^n - c\,(I_{i+1}^n - I_{i-1}^n)/(I_0^n + I_d)$
- $rhs_{max-1}^n = b_2 E_{max-1}^n - aE_{max-2}^n + c\, I_{max-2}^n/(I_{max-1}^n + I_d)$

The tridagonal matrix equation can be solved with an $O(n)$ algorithm, versus $O(n \log_2 n)$ for the spectral method.