1/16/09 Plane wave solutions ①

recall: $$\nabla^2 \vec{E} - \mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

plane wave - no boundary conditions $\Rightarrow$ infinite waves (unphysical)

use separation of variables

$$\vec{E} = \vec{E}_0 E(\vec{r}) E(t)$$

$\vec{E}_0$ has ampl. and polarization $\hat{E}_0$

$|\vec{E}_0| = E_0$

$$\cancel{\vec{E}_0} E(t) \nabla^2 E(\vec{r}) - \mu_0 \varepsilon \cancel{\vec{E}_0} E(\vec{r}) \frac{\partial^2 E(t)}{\partial t^2} = 0$$

$$\frac{\nabla^2 E(\vec{r})}{E(\vec{r})} - \mu_0 \varepsilon \frac{\partial^2 E(t)/\partial t^2}{E(t)} = 0$$

for this eqn. to be valid, each term must be a constant for all space and time

$$\frac{\nabla^2 E(\vec{r})}{E(\vec{r})} = -k^2 \Rightarrow \nabla^2 E(\vec{r}) = -k^2 E(\vec{r})$$

$\hookrightarrow \vec{k} \cdot \vec{k}$

$$\Rightarrow E(\vec{r}) = e^{-j\vec{k}\cdot\vec{r}} \quad (+c.c.)$$

$k = |\vec{k}|$

now substitute into wave equation

$$-k^2 - \mu_0 \varepsilon \frac{\partial^2 E(t)/\partial t^2}{E(t)} = 0$$

ω = radial freq. rad/s

define $\omega^2 = \frac{k^2}{\mu_0 \varepsilon}$

$$\Rightarrow \frac{\partial^2 E(t)}{\partial t^2} = -\frac{k^2}{\mu_0 \varepsilon} E(t)$$

so that $k = \omega \sqrt{\mu_0 \varepsilon}$

wave number rad/m

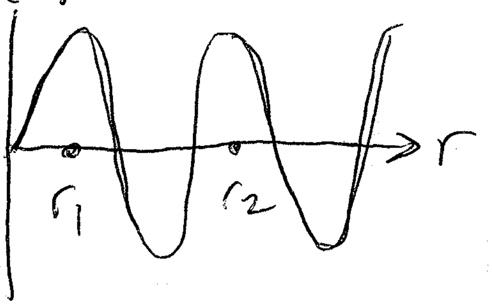
$$= -\omega^2 E(t)$$

$$\Rightarrow E(t) = e^{j\omega t} \quad (+c.c.)$$

$\vec{k}$ = wave vector dir. of phase propagation

$$\vec{E} = \vec{E}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})}$$

~~B~~ $E(r)\;E(t)\quad t=0$



in space, distance between two adjacent peaks is wavelength λ

$$e^{-jkr_1} = e^{-jkr_2}$$
$$= e^{-jk(r_1+\lambda)}$$
$$\Rightarrow e^{-jk\lambda} = 1 \quad \text{or} \quad \lambda = 2\pi/k$$
$$(k = 2\pi/\lambda)$$

can do same for time to get period $T = 2\pi/\omega = 1/\nu$

Transverse waves

assume propagation along z and polarization along x

$$\vec{E}(r,t) = \hat{x} E_0 \cos(\omega t - kz)$$
$$= \frac{\hat{x}}{2} E_0 e^{-j(kz-\omega t)} + c.c. \quad \text{or take real part}$$

find magnetic field

use $\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$$\nabla \times \vec{E} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix}$$
$$= -\hat{y}\left(-\frac{\partial E_x}{\partial z}\right) + \hat{z}\left(-\frac{\partial E_x}{\partial y}\right)^{0}$$

$$\nabla \times \vec{E} = \hat{y}(-jk)E_x$$

only y-component of H - H_y

$$-\frac{jkE_0}{\cancel{2}} e^{-j(kz-\omega t)} \cancel{+c.c.} = -\mu_0 \frac{\partial H_y}{\partial t} \qquad H_y \propto e^{-j(kz-\omega t)} + c.c.$$

$$\frac{jkE_0}{\cancel{2}\mu_0} e^{-j(kz-\omega t)} \cdot \frac{1}{j\omega} \cancel{+c.c.} = H_y$$

$$H_y = \frac{k}{\cancel{2}\mu_0\omega} E_0 e^{-j(kz-\omega t)} + \cancel{c.c.}$$

$\eta_0 = 377\,\Omega$ impedance

$$\frac{k}{\cancel{2}\mu_0\omega} = \frac{\sqrt{\mu_0\epsilon}}{\cancel{2}\mu_0} = \frac{1}{\cancel{2}}\sqrt{\epsilon/\mu_0} = \frac{1}{\cancel{2}\eta}$$

Poynting vector - power flow through surface

$$\vec{S} = \vec{E} \times \vec{H} \quad (W/m^2)$$

$\hat{S}$ = direction of power flow (not necessarily = $\hat{k}$)

using complex notation

$$\langle \vec{S} \rangle = \frac{1}{2} Re[\vec{E}_0 \times \vec{H}_0^*]$$

time average, valid w/o c.c.

$\cancel{Re[\vec{E}_0}$

$$\langle S_z \rangle \; \frac{1}{2} Re[E_0 \times H_0^* \cdot \hat{z}]$$

$$\vec{E}_0 \times \vec{H}_0^* = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_0 & 0 & 0 \\ 0 & \frac{E_0^*}{\eta} & 0 \end{vmatrix} = \hat{z} \frac{|E_0|^2}{\eta}$$

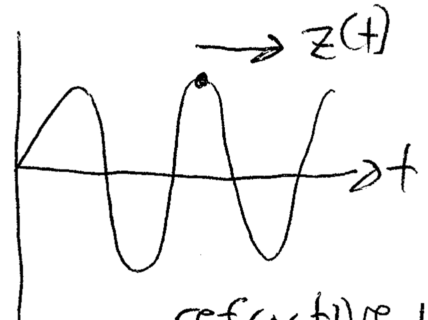
$$\langle S_z \rangle = \frac{1}{2} Re[\checkmark \cdot \hat{z}] = \frac{|E_0|^2}{2\eta}$$

Intensity

phase velocity

$$e^{-j(kz-\omega t)} = \text{const.}$$

$\Rightarrow \; kz - \omega t = \text{const}$



$$z(t) = \text{const} + \omega t / k$$

$$v_p = \frac{dz(t)}{dt} = \frac{\omega}{k} = 1/\sqrt{\mu_0 \epsilon}$$

refractive index

$$n = \frac{c}{v_p} = \sqrt{\frac{\epsilon}{\epsilon_0}}$$

$$k = n k_0 \quad k_0 = \frac{2\pi}{\lambda_0} = \text{free space}$$

group velocity

velocity of envelope of superposition of waves

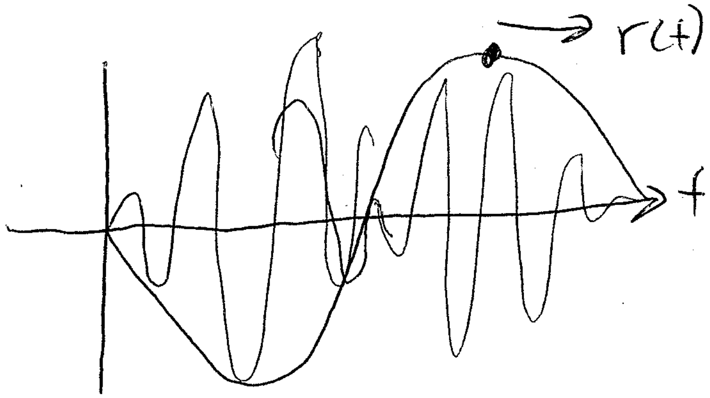
$\omega_1 = \omega + \Delta\omega \qquad \omega_2 = \omega - \Delta\omega$

$k_1 = k + \Delta k \qquad k_2 = k - \Delta k$

$$E_1 + E_2 = E_0 \left\{ \cos\left[(\omega+\Delta\omega)t - (k+\Delta k)z\right] + \cos\left[(\omega-\Delta\omega)t - (k-\Delta k)z\right] \right\}$$

$$\cos(x+y) + \cos(x-y) = 2\cos(x)\cos(y)$$

$$E_1 + E_2 = 2E_0 \cos(\omega t - kz)\cos(\Delta\omega t - \Delta k z)$$



now, envelope velocity

$$z(t) = \frac{\Delta\omega\, t}{\Delta\omega} + \text{const.}$$

$$v_g = \frac{dz(t)}{dt} = \frac{\Delta\omega}{\Delta k} \rightarrow \frac{d\omega}{dk}$$

in free space $\quad v_g = \dfrac{d\omega}{dk} = \dfrac{d(kc)}{dk} = c$

in dispersive medium, i.e. $n(\omega)$ not const.

$$v_g = \frac{d}{dk}\left(\frac{kc}{n}\right) = \frac{c}{n} - \frac{kc}{n^2}\frac{dn}{dk}$$

$$= \frac{c}{n} - \lambda\frac{dn}{d\lambda}$$