waveguide, and so they form stable interference patterns. The structure of the mode is a result of this interference. When the modes constructively interfere, the electric field is maximum, and where destructive interference occurs, the intensity is a minimum.

7.1 Why is β discrete?

The discrete nature of β can be found using this same model of interfering waves. To avoid decay of energy due to destructive interference as the waves travel through the waveguide, the total phase change for a point on the wavefront that travels from one interface ($x = 0$) to the next ($x = -h$), and back again, must be a multiple of 2π. For a wave incident at angle θ, a phase shift of $kn_f h \cos\theta$ is accumulated on the first transverse passage through the film, and a phase shift of $-2\Phi_c$ occurs at the film-cover interface. Another $kn_f h \cos\theta$ of phase is accumulated travelling back down, and finally there is a $-2\Phi_s$ phase shift at the film-substrate interface. The transverse resonance condition requires that

$$2kn_f h \cos\theta - 2\Phi_c - 2\Phi_s = 2\pi\nu \tag{3.32}$$

where ν is an integer. This expression is effectively a dispersion equation for the waveguide. We will use it in the next chapter to develop a generalized dispersion relation for slab waveguides of any construction.

8. Properties of Modes

Once β is determined for a waveguide, the field amplitudes can be described in all regions of the waveguide using Eqs. 3.20 or the equivalent for TM waves. We have been referring to these field distributions as *modes*. The concept of the mode is very powerful — and perhaps a little confusing to the uninitiated. Here we review some of the major properties of modes and modal analysis [5].

The general expression for the electric field solution in all space is

$$\mathbf{E}(x, y, z) = \mathbf{E}(x, y)e^{-j\beta z} \tag{3.33}$$

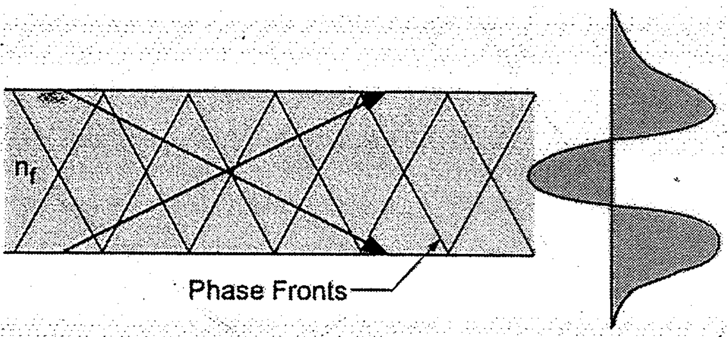

Figure 3.12. A mode can be described as having two plane waves at a slight angle to one another, forming an interference pattern. When the phase fronts cross, there is a maxima.

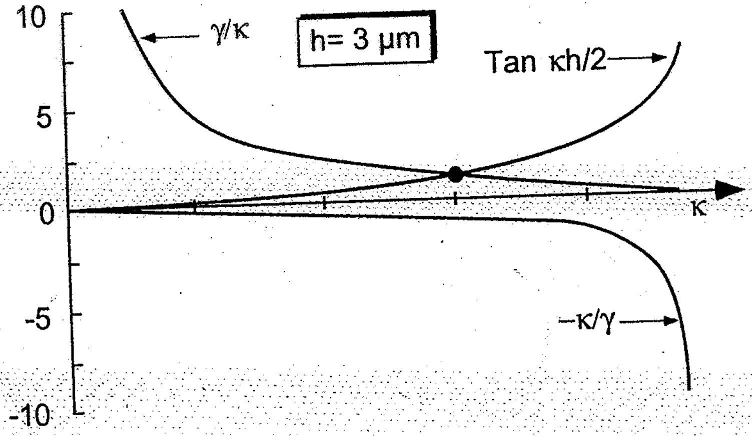

Figure 3.10. For the thin waveguide, there is only one allowed mode, which occurs near $\kappa = 6000 cm^{-1}$.

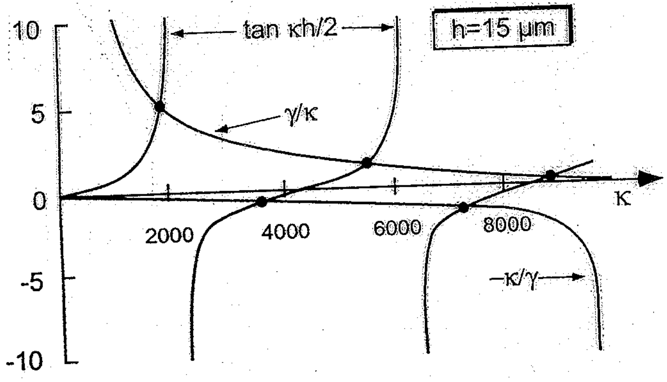

Figure 3.11. The thick waveguide supports both even and odd modes.

the mode structure. Mode structure arises from interference patterns within the waveguide between components of waves travelling in opposite directions. The field within the guiding layer of an even mode in a symmetric waveguide has the form

$$E_y(x) = A \cos \kappa x \, e^{-j\beta z} \tag{3.30}$$

Since $\cos \kappa x = (e^{j\kappa x} + e^{-j\kappa x})/2$, we can rewrite Eq. 3.30 as

$$E_y = \frac{A}{2} \left[e^{+j(\kappa x - \beta z)} + e^{-j(\kappa x + \beta z)} \right] \tag{3.31}$$

Eq. 3.31 represents the superposition of two plane waves, shown schematically in Fig. 3.12. Each plane wave has a k-vector with a transverse component, κ, and a z-component, β. One plane wave has components $\mathbf{k} = \kappa \hat{x} + \beta \hat{z}$, while the other has components $\mathbf{k} = -\kappa \hat{x} + \beta \hat{z}$. Each k-vector has a plane wave associated with it. These two plane waves zig-zag down the waveguide,

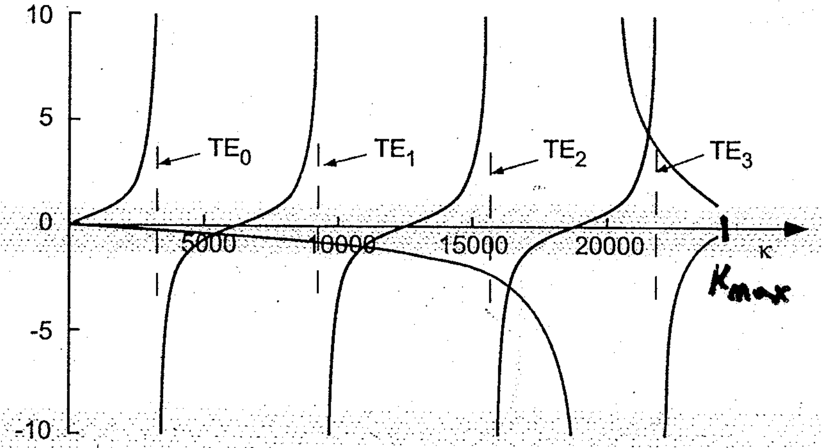

Figure 3.13. The graphical solution to the eigenmode equation for an asymmetric waveguide shows that every time the argument $\kappa_{max}h$ increases by π, another mode is allowed in the waveguide.

κ_{max}. The left-hand-side of Eq. 3.39 is a periodic function ($\tan(\kappa h)$ that goes from $-\infty$ to $+\infty$ every time κh increases by π. Notice that if the value of $\kappa_{max}h$ is greater than $\pi/2$, then we are *guaranteed* to find at least one TE mode in the waveguide. If $\kappa_{max}h > 3\pi/2$, then we are *guaranteed* to find at least two TE modes in the waveguide. These values of $\kappa_{max}h$ are known as *cut-off conditions*. Every time $\kappa_{max}h$ increases by π, another mode is allowed. The approximate number of modes, m, can be found from

$$\begin{aligned} m &= \text{Int}\left[h\kappa_{max}/\pi\right] \\ &= \text{Int}\left[hk(n_f^2 - n_s^2)^{1/2}/\pi\right] \end{aligned} \tag{3.40}$$

This approximation is most accurate when m is a large number. It is approximate because the exact location of the last crossing is not known. Note that the mode count increases with the thickness, h, of the guide, with the difference in index, $(n_f^2 - n_s^2)$, between the core and cladding, and as the wavelength of the guided light gets shorter. Also note that the point at $\kappa = 0$ is not considered to be an allowed mode, even though it appears on the graph that the two equations are crossing at that point.

We usually characterize a waveguide by its *normalized frequency*, defined as

$$V = hk(n_f^2 - n_s^2)^{1/2} \tag{3.41}$$

In terms of the normalized frequency, the approximate number of modes, m, in a waveguide is $m \approx V/\pi$. The mode cut-off conditions are usually described in terms of the normalized frequency. For example, if it is desired to build a waveguide that only carries the first three TE modes, what should the dimen-

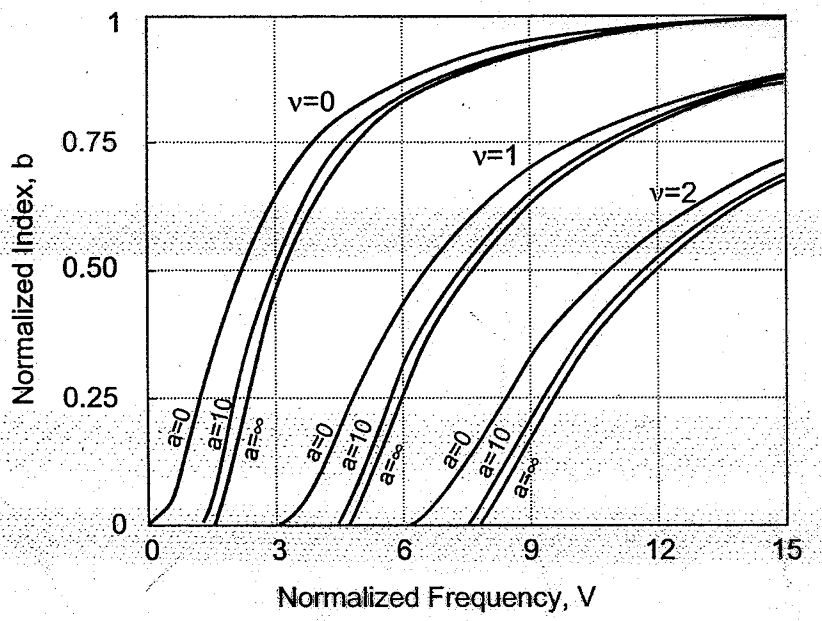

Figure 3.14. The normalized index, b, is plotted against the normalized frequency, V, for three values of the asymmetry coefficient, a, and for the first three values of ν. Values of $a = 0$, $a = 10$, and $a = \infty$ were evaluated.

where ν is an integer. Substituting the normalized parameters into this equation yields the normalized dispersion relation

$$V\sqrt{1-b} = \nu\pi + \tan^{-1}\sqrt{b/(1-b)} + \tan^{-1}\sqrt{(b+a)/(1-b)} \qquad (3.44)$$

At first glance, this may appear as needless complication of straightforward equations, but there is a good reason for this normalization. We can numerically generate a set of curves which relate the normalized index, b, to the normalized frequency, V, using Eq. 3.44. Once the curves are generated, we can relate the calculation to any new waveguide through appropriate scaling. Fig. 3.13 shows the numerically derived relation between the normalized index, b, and the normalized frequency, V.

Example 3.3 Evaluation of a waveguide using normalized parameters

To illustrate the power of using the normalized parameters, let's reconsider the waveguide used in Example 3.1, which had a guiding index, $n_f = 1.5$, a substrate index $n_s = 1.45$, a cover index $n_c = 1.40$, a film thickness $h = 5\mu m$, and a driving wavelength of $\lambda_0 = 1\mu m$. Using a numerical solution, we found the eigenvalues for the first three modes to be $\beta = 94087$, 93608, and 92819 cm^{-1}. We can determine the propagation coefficients by inspection using the graph in Fig. 3.14. First we must normalize the waveguide parameters,

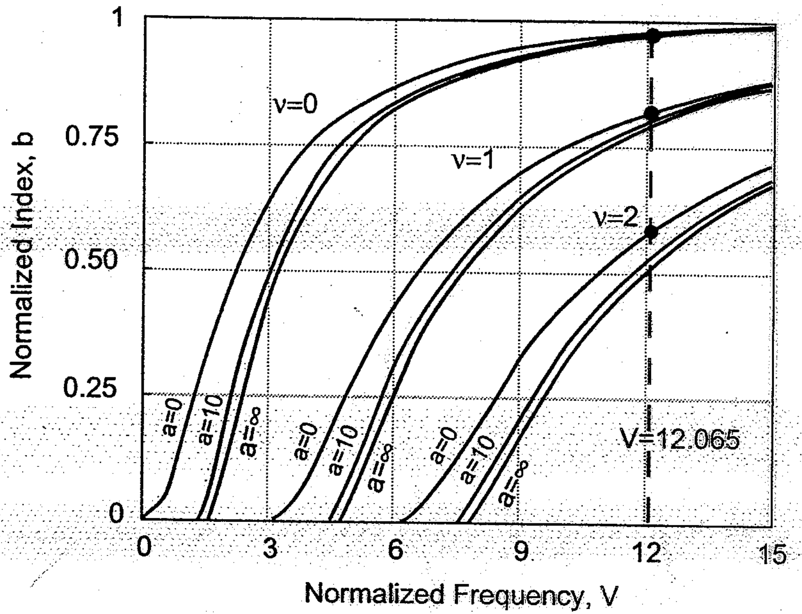

Figure 3.15. Data from Example 3.2 plotted on graph from Fig. 3.14.

$$a = (n_s^2 - n_c^2)/(n_f^2 - n_s^2) = (1.45^2 - 1.40^2)/(1.55^2 - 1.45^2) = 0.475$$

The asymmetry value $a = 0.475$ lies near the $a = 0$ value of Fig. 3.13, so we will interpolate between the two plotted lines for each value of ν in the plot. We draw a line on the graph at V=12.065, as shown in Fig. 3.14, and read the b values from the scale.

At a normalized frequency of $V = 12.065$ there are three values of b: 0.575, 0.813, and 0.965. (Note: the fourth mode that we found in Example 3.1 is not found here because the graph in Fig. 3.14 does not show a curve for the $\nu = 3$ case. The vertical line in Fig. 3.15 shows the intersections. Using the expression for b

$$b = (n_{eff}^2 - n_s^2)/(n_f^2 - n_s^2)$$

we can solve this for β noting that $n_{eff} = \beta/k_0$

$$\beta = k_0\sqrt{(n_f^2 - n_s^2)b + n_s^2}$$

Plugging values into the equation, we get the first three allowed values of β. These are tabulated alongside the "exact" values obtained by numeric technique for comparison.

The agreement is remarkable: better than 1 part in a 1000, which can be attributed to ones ability to read the graph in Fig. 3.13. The