A Complete Characterization of Torque Maximization for Permanent Magnet Non-Salient Synchronous Motors

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Abstract

This paper considers the problem of maximizing the torque of permanent magnet non-salient synchronous motors in the presence of voltage and current constraints. Formulas are given that enable a drive to extract the maximum torque available from a motor in real-time. Both the motoring and braking modes are considered, and it is shown that the well-known field-weakening formula cannot be used with some motors. An example is discussed to illustrate the contributions of the paper.

1 Introduction

Maximization of the torque produced by electric motors is an important practical consideration, since optimization may yield the use of a smaller motor for a given application, or higher performance for a given motor. In this paper, we consider the problem of extracting all the available torque from permanent magnet, non-salient synchronous motors. These motors include brushless DC motors and stepper motors, and are commonly used in robotic applications. Another emerging application area is in electric, or hybrid-electric vehicles, where torque optimization is useful to keep the size and cost of the motor down.

The available torque of electric motors decreases at high speed because the source voltage must overcome the increasing back-emf voltage at the detriment of building up the stator currents. Maximization of the torque requires the use of field-weakening which, for synchronous motors, may be achieved optimally using a relatively simple strategy. The relevant formula is derived assuming that the speed is constant, an assumption that is not satisfied in practice. However, the speed usually varies sufficiently slowly compared to the electrical dynamics that the formula is useful [1].

An interesting question is how one may transition between the field-weakening solution that is optimal at high speed and the low speed solution, which is determined by current limits rather than voltage limits. At low speed, torque maximization is achieved by aligning the stator current vector (which produces the stator magnetic field) with the direction perpendicular to the rotor's permanent magnet, a strategy which is usually referred to as quadrature control. In [6], it was shown that transitioning between the two solutions could naturally be achieved by considering the optimal control problem under joint voltage and current constraints over the whole speed range. The resulting strategy involved the two classical solutions (quadrature control at low speed and field-weakening at high speed), and an intermediate region where both the voltage and the current limits were active. The paper provided algorithms to compute the speeds at which transitions occurred between the three modes, and described the control strategies required in each case. Experiments were reported that demonstrated the use of the techniques on a stepper motor.

In this paper, we expand the results of [6] in two ways. First, we consider the case of braking (negative torque for positive speed), as well as motoring. Although differences between the two cases are small for large motors, the optimal solutions are numerically different, and noticeably so for small motors. Second, we consider cases that were not investigated before, being considered unlikely to occur in practice. Although this assumption was reasonable for stepper motors, examples of brushless DC motors were found that fall into this unexpected category. A surprising by-product of the analysis is that some motors never reach the second transition speed, *i.e.,* that torque optimization at high speed is *always* constrained by both the current and voltage limits. In such cases, the traditional field weakening formula cannot be applied, and must be replaced by the formula provided in this paper. An example is provided to demonstrate the application of the results in a practical problem.

2 Model of the Motor and Problem Statement

Considering the application to a brushless DC motor, we begin with the model of a three-phase synchronous motor, as given in [2]

$$L_S\frac{di_{S1}}{dt} - M\frac{di_{S2}}{dt} - M\frac{di_{S3}}{dt} = v_{S1} - Ri_{S1} + K_m\omega\sin(n_p\theta) \tag{1}$$

$$-M\frac{di_{S1}}{dt} + L_S\frac{di_{S2}}{dt} - M\frac{di_{S3}}{dt} = v_{S2} - Ri_{S2} + K_m\omega\sin(n_p\theta - \frac{2\pi}{3}) \tag{2}$$

$$-M\frac{di_{S1}}{dt} - M\frac{di_{S2}}{dt} + L_S\frac{di_{S3}}{dt} = v_{S3} - Ri_{S3} + K_m\omega\sin(n_p\theta - \frac{4\pi}{3}) \tag{3}$$

$$J\frac{d\omega}{dt} = -K_m i_{S1}\sin(n_p\theta) - K_m i_{S2}\sin(n_p\theta - \frac{2\pi}{3}) - K_m i_{S3}\sin(n_p\theta - \frac{4\pi}{3}) - \tau_L \tag{4}$$

L_S is the self-inductance of a stator winding, M is the coefficient of mutual inductance between the phases, K_m is the torque/back-emf constant, R is the resistance of a stator winding, J is the moment of inertia of the rotor, τ_L is the load torque, θ is the rotor angular position, ω is the rotor speed and n_p is the number of pole pairs (or the number of rotor teeth for a stepper motor). If the phases were perfectly coupled, one would have $M = L_S/2$.

The power-preserving *three-phase to two-phase transformation* is defined by

$$\begin{bmatrix} i_a \\ i_b \\ i_0 \end{bmatrix} \triangleq \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} v_a \\ v_b \\ v_0 \end{bmatrix} \triangleq \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} v_{S1} \\ v_{S2} \\ v_{S3} \end{bmatrix} \tag{6}$$

and transforms the original model into the equivalent model

$$(L_S + M)\frac{di_a}{dt} = v_a - Ri_a + \sqrt{\frac{3}{2}}K_m\omega\sin(n_p\theta) \tag{7}$$

$$(L_S + M)\frac{di_b}{dt} = v_b - Ri_b - \sqrt{\frac{3}{2}}K_m\omega\cos(n_p\theta) \tag{8}$$

$$(L_S - 2M)\frac{di_0}{dt} = \frac{1}{\sqrt{3}}v_0 - \frac{1}{\sqrt{3}}Ri_0 \tag{9}$$

$$J\frac{d\omega}{dt} = -\sqrt{\frac{3}{2}}K_m i_a\sin(n_p\theta) + \sqrt{\frac{3}{2}}K_m i_b\cos(n_p\theta) - \tau_L \tag{10}$$

For a balanced three-phase system in which $v_0 = v_{S1} + v_{S2} + v_{S3} = 0$, $i_0 = i_{S1} + i_{S2} + i_{S3} = 0$, one obtains the *two-phase equivalent model* given by

$$L\frac{di_a}{dt} = v_a - Ri_a + K\omega\sin(n_p\theta) \tag{11}$$

$$L\frac{di_b}{dt} = v_b - Ri_b - K\omega\cos(n_p\theta) \tag{12}$$

$$J\frac{d\omega}{dt} = -Ki_a\sin(n_p\theta) + Ki_b\cos(n_p\theta) - \tau_L \tag{13}$$

where $L = L_S + M$ (approximately $3L_S/2$), $K = \sqrt{3/2}K_m$, i_a and i_b are the equivalent currents in phases a and b, respectively.

Letting V_{bus} denote the DC bus voltage of a 3-phase inverter, the maximum voltage out of the inverter is obtained in six step mode, resulting in a peak of the fundamental waveform equal to $v_{\max} = (2/\pi)V_{bus}$, which is the maximum phase voltage. With $i_{\max}$ and $v_{\max}$ denoting the limits of the 3-phase currents and voltages, respectively, the corresponding limits $I_{\max}, V_{\max}$ for the equivalent 2-phase motor are

$$I_{\max} = \sqrt{\frac{3}{2}}i_{\max} \tag{14}$$

$$V_{\max} = \sqrt{\frac{3}{2}}v_{\max} = \sqrt{\frac{3}{2}}\frac{2}{\pi}V_{bus} \tag{15}$$

The *direct-quadrature* or *dq* transformation is useful to analyze the properties of the motor and is given by

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} i_a \\ i_b \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} v_d \\ v_q \end{bmatrix} = \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} \tag{17}$$

where i_d, i_q and v_d, v_q are the transformed currents and voltages, respectively, in the dq (for direct and quadrature) reference frame. The state-space model in the dq coordinates is

$$L\frac{di_d}{dt} = v_d - Ri_d + n_p\omega Li_q \tag{18}$$

$$L\frac{di_q}{dt} = v_q - Ri_q - n_p\omega Li_d - K\omega \tag{19}$$

$$J\frac{d\omega}{dt} = Ki_q - \tau_L \tag{20}$$

$$\frac{d\theta}{dt} = \omega. \tag{21}$$

This model assumes that the rotor is smooth (non-salient) and that the magnetics are linear. For constant speed ω, equations (18) to (20) yield

$$v_d = Ri_d - n_p\omega Li_q \tag{22}$$

$$v_q = Ri_q + n_p\omega Li_d + K\omega \tag{23}$$

$$Ki_q = \tau_L. \tag{24}$$

The problem under consideration is the maximization of the electrical torque Ki_q within the voltage and current constraints. The voltage and current constraints are incorporated into the problem statement by using the fact that, at constant speed, the phase voltages and the phase currents are sinusoidal, with peak magnitudes given by $\sqrt{v_d^2 + v_q^2}$ and $\sqrt{i_d^2 + i_q^2}$ respectively. Therefore, the phase voltages bounded by V_{max} and the phase currents bounded by I_{max}, yield constraints in the DQ variables given by

$$V = \sqrt{v_d^2 + v_q^2} \le V_{max} \tag{25}$$

$$I = \sqrt{i_d^2 + i_q^2} \le I_{max}. \tag{26}$$

3 Torque Maximization

3.1 Speed Ranges and Transition Speeds

Because the torque may, in theory, be made arbitrarily large by increasing the voltage and current levels, optimal operation under constraints is always achieved with the limits on (either or both) the currents and voltages being reached. A typical motor is characterized by three ranges of speed: a low-speed range where operation is constrained by the current limit only, an intermediate speed range where operation is constrained by both the voltage and the current limit, and a high-speed range where operation is constrained by the voltage limit only. If the current limit is very high ($I_{max} > V_{max}/R$), it is possible for the whole speed range to be constrained by the voltage limit alone. However, this is not typically the case.

At a specific speed, torque maximization will be achieved for a specific choice of i_d and i_q (for a current-controlled drive), or a specific choice of v_d and v_q (for a voltage-controlled drive). The optimal values are denoted by i_d^*, i_q^*, v_d^*, and v_q^*. Knowing what the solution is in each of the speed ranges, it remains to determine what the boundaries of those ranges are in order to determine which solution is to be applied. The two speeds where one transitions from one type of constraint to another are referred to as *the first and second transition speeds*, and denoted as ω_1 and ω_2, repectively. These speeds can be calculated after having obtained the form of the optimal solution in each speed range. In the analysis that follows, only the case $\omega \geq 0$ will be considered because straightforward symmetry arguments will easily yield the case $\omega \leq 0$.

3.2 Case 1: Maximization without the Voltage Constraint

First consider the case where only the current constraint is active. In this case, the optimum i_d^* must be zero to maximize the current i_q. The overall solution is simply

$$i_d^* = 0 \tag{27}$$

$$i_q^* = \pm I_{max} \tag{28}$$

$$v_d^* = \pm(-n_p \omega L I_{max}) \tag{29}$$

$$v_q^* = \pm R I_{max} + K\omega. \tag{30}$$

where the positive sign is chosen for positive torque and the negative sign is chosen for negative torque.

3.3 Case 2: Maximization without the Current Constraint

Torque maximization under the voltage constraint alone yields a solution which is referred to as *optimal field-weakening* and is well-known (*cf.* [2], p. 266.). To obtain the solution, one solves (22) and (23) for i_d and i_q, so that

$$\begin{pmatrix} i_d \\ i_q \end{pmatrix} = \frac{1}{Z} \begin{pmatrix} R v_d + n_p \omega L (v_q - K\omega) \\ -n_p \omega L v_d + R(v_q - K\omega) \end{pmatrix}, \tag{31}$$

where $Z = R^2 + (n_p \omega L)^2$. If one defines V and α so that $v_d = V\cos(\alpha), v_q = V\sin(\alpha)$, the current i_q is given by

$$i_q = \frac{-n_p \omega L V \cos(\alpha) + RV\sin(\alpha) - K\omega R}{Z}. \tag{32}$$

Equation (32) shows that the maximum current i_q, and therefore the maximum electrical torque, is obtained for

$$V = \pm V_{max}$$

$$\frac{d}{d\alpha}\left(-n_p \omega L \cos(\alpha) + R\sin(\alpha)\right) = 0. \tag{33}$$

where the positive sign is chosen for positive torque and the negative sign is chosen for negative torque. Solving the second of these equations gives $\tan(\alpha^*) = -R/(n_p \omega L)$. The optimal v_d^* and v_q^* therefore satisfy

$$\frac{v_d^*}{v_q^*} = \frac{-n_p \omega L}{R} = \frac{1}{\tan(\alpha^*)}. \tag{34}$$

This equation gives the value of the *optimal lead angle* for voltage-controlled drives. Equation (34) yields $\sin(\alpha^*) = R/\sqrt{Z}, \cos(\alpha^*) = -n_p\omega L/\sqrt{Z}$ and therefore

$$v_d^* = \pm\frac{-n_p\omega L}{\sqrt{Z}} V_{max} \tag{35}$$

$$v_q^* = \pm\frac{R}{\sqrt{Z}} V_{max}. \tag{36}$$

For a current-controlled drive, one finds i_d^*, using (31) and (34), so that

$$i_d^* = -\frac{(n_p\omega L)(K\omega)}{Z}, \tag{37}$$

which can be viewed as the *optimal field weakening current.* Using (32) and the above expressions for $\sin(\alpha^*), \cos(\alpha^*)$ with $V = \pm V_{max}$ results in

$$i_q^* = \frac{\pm V_{max}\sqrt{Z} - K\omega R}{Z}. \tag{38}$$

Note that if $R = 0$, then the motoring and braking quadrature currents are the negative of each other.

3.4 Case 3: Maximization under Voltage and Current Constraints

In the case where both the voltage and the current constraints are active, the optimization reduces to the solution of a system of algebraic equations, because there are as many constraints as degrees of freedom. The steady-state equations (22), (23), with $v_d^2 + v_q^2 = V_{max}^2$, yield

$$Z(i_d^2 + i_q^2) + K^2\omega^2 + 2RK\omega i_q + 2n_pLK\omega^2 i_d = V_{max}^2. \tag{39}$$

Eliminating i_d using $i_d^2 + i_q^2 = I_{max}^2$ leads to a quadratic equation for i_q^* of the form

$$\alpha i_q^{*2} + \beta i_q^* + \gamma = 0, \tag{40}$$

with

$$\alpha = 4R^2\omega^2K^2 + 4n_p^2\omega^4L^2K^2 \tag{41}$$

$$\beta = -4RK\omega(V_{max}^2 - K^2\omega^2 - ZI_{max}^2) \tag{42}$$

$$\gamma = (V_{max}^2 - K^2\omega^2 - ZI_{max}^2)^2 - 4I_{max}^2 n_p^2\omega^4L^2K^2. \tag{43}$$

In this case (where both the voltage and the current constraints are active), $\beta^2 - 4\alpha\gamma \geq 0$, and equation (40) has two real roots

$$i_q^* = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \tag{44}$$

with the positive sign for motoring and the negative sign for braking. The other corresponding variables can be determined from i_q^* using

$$\begin{aligned} i_d^* &= -\sqrt{I_{max}^2 - (i_q^*)^2} && (45)\\ v_d^* &= Ri_d^* - n_p\omega L i_q^* && (46)\\ v_q^* &= Ri_q^* + n_p\omega L i_d^* + K\omega. && (47) \end{aligned}$$

Again, note that if $R = 0$ so that $\beta = 0$, the motoring and braking quadrature currents are just the negative of each other.

3.5 First Transition Speed ω_1

The first transition speed ω_1 can be found by calculating the value of the speed for which the voltage limit is reached under the optimal solution obtained with only the current constraint. With $v_d^2 + v_q^2 = V_{max}^2$, (29) and (30) yield a quadratic equation for the first transition speed

$$(K^2 + (n_p L I_{max})^2)\omega_1^2 + 2RKI_{max}\omega_1 = V_{max}^2. \tag{48}$$

The equation has two real roots $\omega_{1pos} > 0$ and $\omega_{1neg} < 0$. The positive root ω_{1pos} corresponds to the case of positive torque. The negative root corresponds to the braking mode, that is, $|\omega_{1neg}|$ is the transition speed when $i_q = -I_{\max}$. For $R = 0$, $\omega_{1neg} = -\omega_{1pos}$.

3.6 Second Transition Speed ω_2

The second transition speed ω_2 can be calculated by finding the value of the speed such that the optimum solution obtained with the voltage constraint alone reaches the current limit. We will show that a cubic equation can be obtained, so that the solution can be calculated exactly using standard formulas. Using (37), (38), and $i_d^2 + i_q^2 = I_{max}^2$, one first finds that

$$Z\left(V_{max}^2 + (K\omega)^2 - ZI_{max}^2\right) = \pm 2K\omega R V_{max}\sqrt{Z} \tag{49}$$

and, cancelling the square-root term and squaring both sides,

$$Z\left((V_{max}^2 - R^2 I_{max}^2) - (n_p^2 L^2 I_{max}^2 - K^2)\omega^2\right)^2 = (2KRV_{max})^2\omega^2. \tag{50}$$

A cubic equation is obtained for ω_2^2, of the form

$$(x-a)^2(x+b) = cx, \tag{51}$$

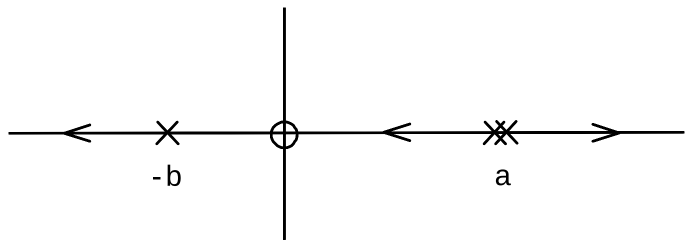

Figure 1: Complementary root-locus when a > 0

where

$$x = \omega_2^2 \tag{52}$$
$$a = \frac{V_{max}^2 - R^2 I_{max}^2}{n_p^2 L^2 I_{max}^2 - K^2} \tag{53}$$
$$b = \frac{R^2}{(n_p L)^2} \tag{54}$$
$$c = \frac{(2KRV_{max})^2}{(n_p L)^2 (n_p^2 L^2 I_{max}^2 - K^2)^2}. \tag{55}$$

To study the roots of equation (51), we rearrange it as

$$1 - c\frac{x}{(x-a)^2(x+b)} = 0 \tag{56}$$

and compute the root locations using the properties of the *complementary root-locus*, *i.e.*, the root-locus corresponding to a zero at $x = 0$, poles at $x = -b$ and $x = a$ (double pole), and a *negative* gain $k = -c$. This analysis is now broken down into two cases.

3.6.1 Case 1: $a > 0$

The most common situation is the one in which $a > 0$. The quantity $V_{max}^2 - R^2 I_{max}^2 > 0$ for any practical motor, so that the case $a > 0$ actually implies that $n_p^2 L^2 I_{max}^2 - K^2 > 0$. In this case, the complementary root-locus is shown in Fig. 1. For $c > 0$, there is one negative root $x < -b$, a positive root $x < a$, and a positive root $x > a$. As $x = \omega_2^2$, the root $x < -b$ is extraneous and therefore ignored. To understand which of the other two roots may apply, we return to the original equation (49) and rearrange it as

$$Z\left(n_p^2 L^2 I_{max}^2 - K^2\right)\left(\frac{V_{max}^2 - R^2 I_{max}^2}{n_p^2 L^2 I_{max}^2 - K^2} - \omega^2\right) = \pm 2K\omega R V_{max}\sqrt{Z} \tag{57}$$

This expression shows that the root satisfying $x < a$ corresponds to positive torque ($+V_{max}$) while the root satisfying $x > a$ corresponds to the negative torque ($-V_{max}$).

3.6.2 Case 2: $a < 0$

The solutions of (56), or equivalently (51), when $a < 0$ are now considered. This case requires $K > n_p L I_{max}$ which is unlikely to occur in stepper motors because of the large number of pole pairs. Therefore, it was not investigated in [6]. However, we will show an example of a brushless DC motor that falls into this category. We will show that in the case $a < 0$, there is never a second transition speed in brake mode, while in the motor mode, there is either *no* second transition speed, or *two* second transition speeds (that is, *two* speeds $\omega_{21}$and ω_{22} where the system transitions between the $V = V_{\max}$ constraint region and the $V = V_{\max}, I = I_{\max}$ constraint region)

In the case $a < 0$, the complementary root-locus is shown in Fig. 2. The break-away points are the roots of

$$\frac{dc}{dx} = \frac{d}{dx}\frac{(x-a)^2(b+x)}{x} = \frac{(x-a)(2x^2+bx+ab)}{x^2} = 0. \tag{58}$$

The two solutions of this equation that belong to the complementary root-locus are

$$\begin{aligned} x_b &= a \\ x_b &= \frac{-b+\sqrt{b^2-8ab}}{4}. \end{aligned} \tag{59}$$

The second root is the break-away point on the positive real axis. For roots of (56) to be real and positive, the given value of c must be greater than or equal to the value corresponding to the break-away point. In other words, there are two positive roots x_{21}, x_{22} of (56) satisfying $x_{21} < x_b$ and $x_{22} > x_b$ if and only if

$$1 - c\frac{x_b}{(x_b-a)^2(b+x_b)} > 0. \tag{60}$$

Note that, as $a < 0$ (so that $n_p^2 L^2 I_{max}^2 - K^2 < 0$), both $x_{21}$and x_{22} satisfy (57), or equivalently (49), with $+$ sign chosen (i.e., $+V_{\max}$) rather than the $-$ sign (i.e., $-V_{\max}$). In other words, *both* transition speeds $\omega_{21} = \sqrt{x_{21}}, \omega_{22} = \sqrt{x_{22}}$ are associated with the motoring mode (as opposed to the case $a > 0$ where they were split between the two modes). As a result, there is no second transition speed in the brake mode. In motor mode, there are actually *four* speed ranges in this case. They are, in order of increasing speed, constrained by: (1) current limits; (2) current and voltage limits; (3) voltage limits; (4) current and voltage limits.

Although hypothetical motor parameters can be selected to obtain the case with $a < 0$ and two second transition speeds, they do not correspond to realistic motors. The typical case is the one in which (60) is not satisfied, and there is no second transition speed in either motor or brake mode. Then, there are only speed ranges for torque optimization: a low speed range constrained by current limits, and a high speed range constrained by voltage and current limits.

4 Simulation Results

The motor chosen for consideration is the BM 500 from Aerotech Inc. (Pittsburgh, PA). Its parameter values are given by $L_S = 9.3 \times 10^{-4}$ H, $M = L_S/2$ H, $R = .25$ Ohms, $J = 13.9 \times 10^{-5}$ N-m-s^2, $K_m = 0.198$ N-m/Amps, $n_p = 4$. The limits are given by $i_{\max}$ (continuous) $= 18$ Amps, $i_{\max}$ (peak) $= 55$ Amps, and V_{bus} (bus voltage) $= 160$ Volts. The motor bearings are rated for $10,000$ rpm or about 1000 rads/sec. We

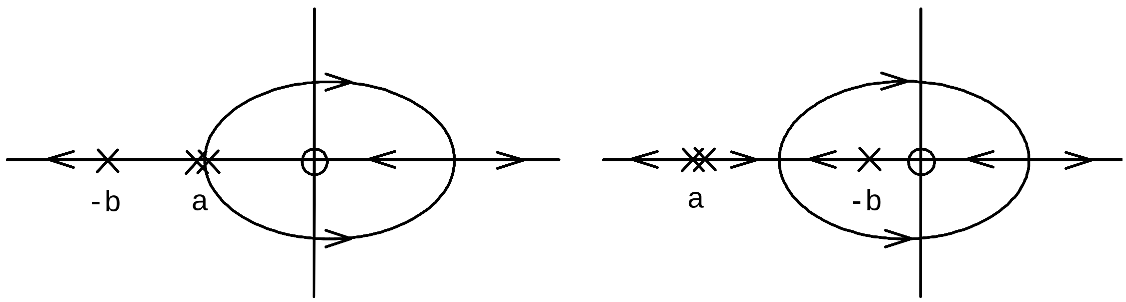

Figure 2: Complementary root-locus when a < 0

consider this to be the maximum speed of the motor. The corresponding equivalent two-phase parameters are $L = L_S + M = 14 \times 10^{-4}$ H, $K = (\sqrt{3/2})K_m = .243$ N-m/Amps, $I_{\max}(\text{continuous}) = (\sqrt{3/2})i_{\max}$ (continuous) $= 22$ Amps, $I_{\max}$ (peak) $= \sqrt{3/2}i_{\max}$ (peak) $= 67.4$ Amps, $V_{\max} = \sqrt{3/2}(2/\pi)V_{bus} = 124.8$ Volts.

In Figure 3 are plotted the component torques for $I_{\max} = 22$ Amps (i.e., the maximum torque for the three cases (1) $I = I_{\max}$ (2) $V = V_{\max}$, and (3) $I = I_{\max}, V = V_{\max}$) as a function of the speed ω. Both positive and negative torques are given. The curves give the torques obtained by the three strategies, accounting for the voltage and current limits. The actual maximum torque attainable is the largest of the three curves at each speed (the outer envelope of the curves).

In this case, the first transition speed for the motor mode is 592 rads/sec, while the first transition speed for the brake mode is 635 is rads/sec. There are no second transition speeds. As one can see from the graph, the reason for this situation is that the torque curve obtained with optimal field-weakening ($V = V_{\max}$) falls rapidly due to the current limit. The torque is zero at the speed where the optimal field-weakening current is equal to the maximum current, leaving no torque-producing current. For this reason, a solution accounting for both current and voltage limits must be used at high speed.

In Figure 4, the torque curves are plotted as functions of the speed ω for $I_{\max} = 67.4$ Amps (peak current limit). Both positive and negative torques are given. Again, the actual maximum torque is simply the maximum of the three curves at each speed. In this case, the first transition speed for the motor mode is 288 rads/sec while the second transition speed for the motor mode is 320.6 rads/sec. The first transition speed for the brake mode is 340.8 rads/sec while the second transition speed for the brake mode is 383 rads/sec.

The above analysis assumed constant speed, steady-state conditions. To see how this approach works under non-constant speed conditions, a time-domain simulation was performed. Figures 5, 6 and 7 show the results of the simulation of the motor with $I_{\max} = 22$ Amp (continuous current limit), $V_{\max} = 124.8$ Volts where the motor is driven by the maximum torque (Ki_q^*) until the speed reaches approximately the maximum speed 1000 rads/sec ($10,000$ rpm or the rated speed of the motor bearings). This result was obtained by choosing for the current commands the optimal values i_d^*, i_q^*, which are plotted as $i_d^*/I_{\max}, i_q^*/I_{\max}$ in Figure 5.

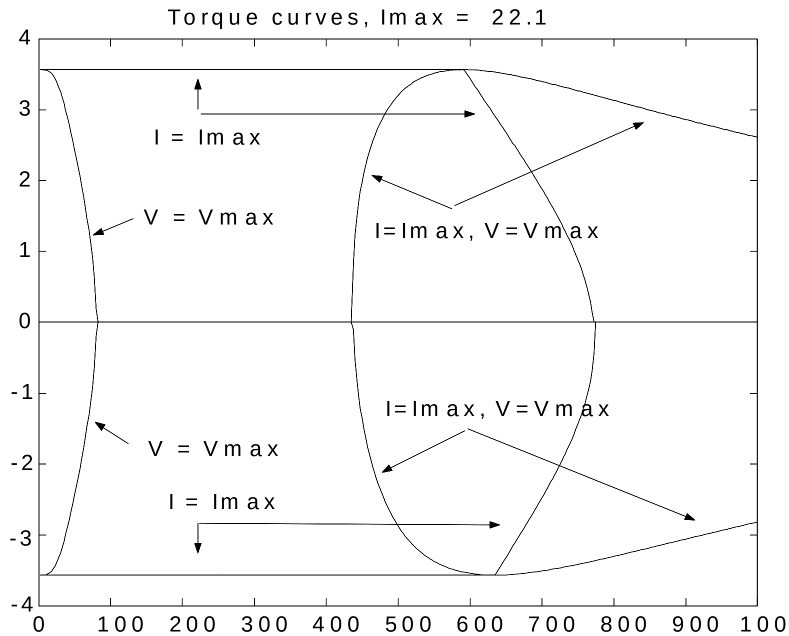

Figure 3: Torque curves associated with the voltage and continuous current limits

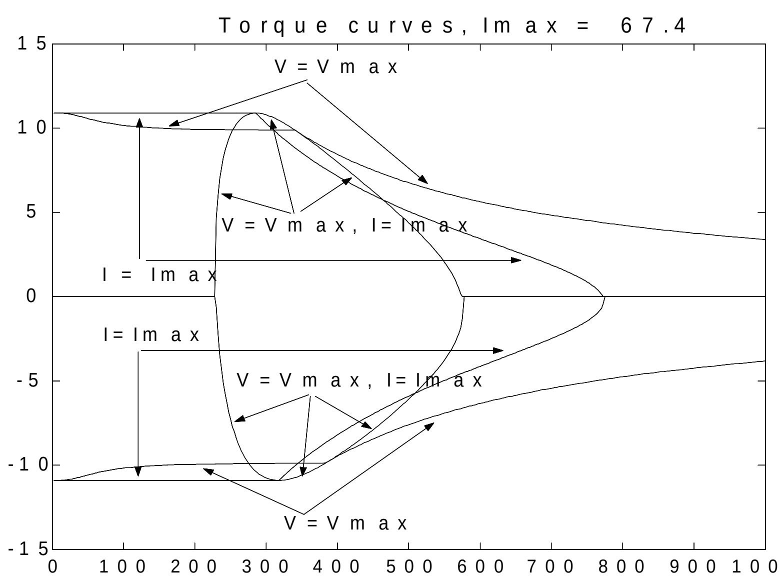

Figure 4: Torque curves associated with the voltage and peak current limits

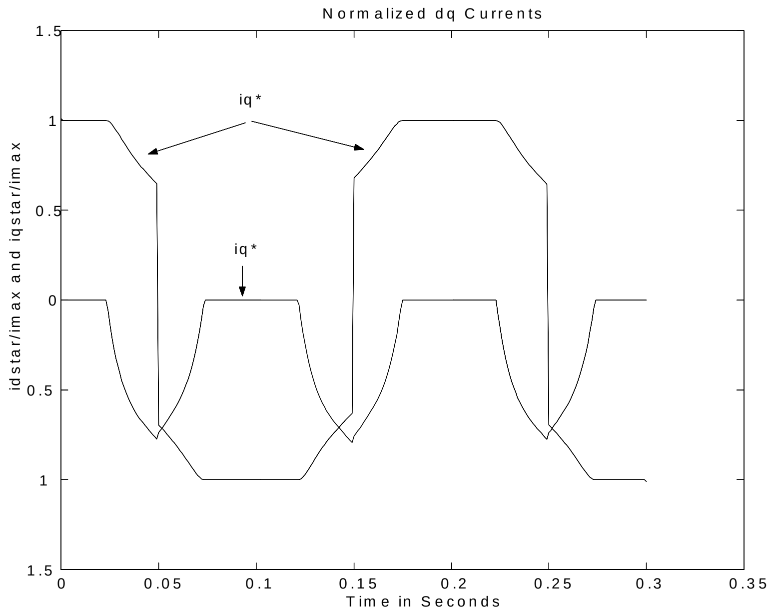

Figure 5: Optimal direct and quadtrature currents

Figure 6 shows that the motor reaches the speed of 1000 rads/sec in about 40 milliseconds. The normalized magnitude of the *commanded* voltage $V_{mag}/V_{\max}$ (with $V_{mag} = \sqrt{v_d^2 + v_q^2}$) and current $I_{mag}/I_{\max}$ (with $I_{mag} = \sqrt{i_d^2 + i_q^2}$) are shown in Figure 7. Note that the commanded voltage has sharp peaks which are greater than the normalized limit of 1 at the times when the commanded current i_q^* undergoes step changes from the maximum positive value to the maximum negative value (or vice-versa). The PI current controller given by $v_q = K_p(i_q^* - i_q) + K_I \int_0^t (i_q^* - i_q)dt$ is commanding a large voltage due to the large step change in the error $i_q^* - i_q$ at these points in time. However, the voltage actually applied to the motor in the simulation was limited to $V_{\max}$, indicating that these commanded overvoltage peaks have no effect on the performance.

5 Conclusions

The paper has presented a complete characterization of the dq currents and voltages required to achieve the maximum torque of permanent magnet non-salient synchronous motors. A useful feature of the solution is that it consists of explicit, analytic formulas. In fact, given that the most complicated calculations are square roots and cubic roots, the whole set of computations could be performed in real-time if the motor parameters were determined on-line, using an identification procedure such as described in [3].

In general, there is an intermediate speed range where both the voltage and the current limits are reached. Consideration of this speed range not only provides the complete solution to the problem, but also provides a smooth connection between the solutions for the low and high speed regimes. Further, there are cases where

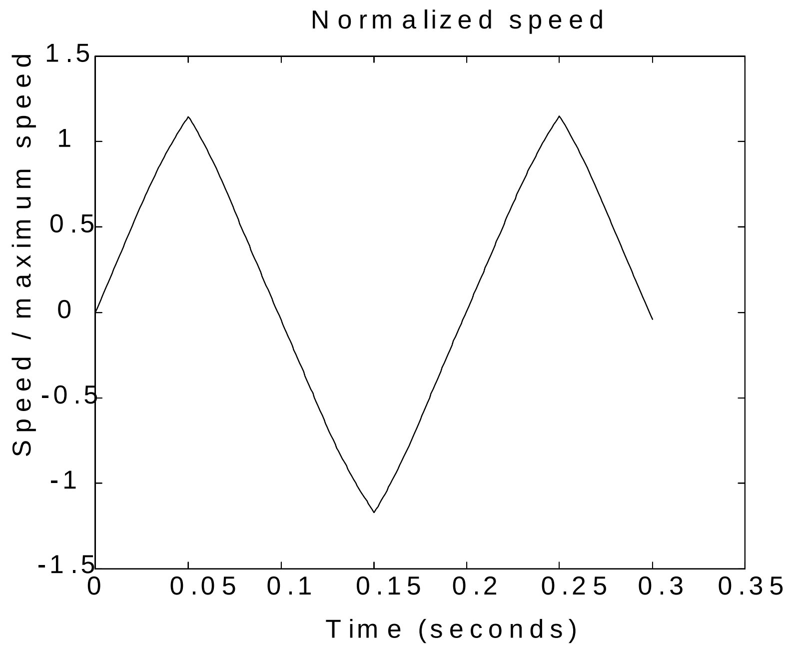

Figure 6: Speed response

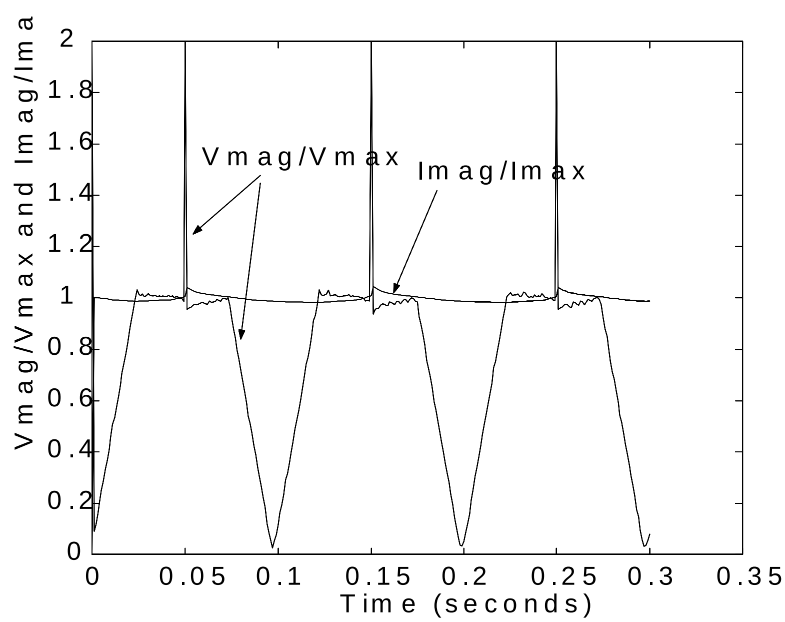

Figure 7: Normalized magnitude of the current and commanded voltage

the optimal operation at high speed is determined solely by the joint voltage/current limit solution. In such cases, the conventional optimal field weakening formula never provides the optimum torque.

The presence of speed ranges where both the voltage and the current limits were active was also observed in induction motors (see [4]). However, no analytic expression could be found in that case, because of higher-order polynomial equations. A substantial difference existed in induction motors between the motoring and generating mode. For synchronous motors, the difference between the motor and braking cases is not so great, but can also be observed. For $R = 0$, the difference disappears and one simply switches the sign of the current i_q, leaving i_d identical.

Note that another approach to the torque optimization problem was recently proposed by Lawler *et al.* [5]. In their approach, the phase advance method was modified to handle the intermediate region (*i.e.,* when the system is at both the voltage and current constraints) by a new inverter topology. In contrast, the approach here uses a standard inverter topology.

6 Acknowledgments

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7 Appendix B Procedure for computing the currents in Figures 3 & 4

7.1 Case 1

Here $i_d^* = 0, i_q^* = \pm I_{max}$ and the magnitude of the voltage required to achieve the maximum current I_{max} must satisfy

$$\begin{aligned}(v_d^*)^2 + (v_q^*)^2 &= (-n_p \omega L I_{max})^2 + (R I_{max} + K\omega)^2 \\ &= Z I_{max}^2 + 2RK\omega I_{\max} + K^2\omega^2 \leq V_{\max}^2\end{aligned}$$

For higher speeds, this constraint cannot be satisfied and the current i_q^* must be reduced below $I_{\max}$. Setting $i_q^* = I$, the maximum current available for i_q^* is found by solving

$$Z I^2 + 2RK\omega I + K^2\omega^2 = V_{\max}^2$$

Solving,

$$I = \frac{-RK\omega \pm \sqrt{(RK\omega)^2 + V_{\max}^2 - Z(K\omega)^2}}{Z}$$

where the + sign is for motoring and the − sign is for braking. In summary, with $i_d^* = 0$ the direct current reference and

$$\begin{aligned}i_q^* &= \pm I_{\max} \text{ for } (v_d^*)^2 + (v_q^*)^2 \leq V_{\max}^2 \\ &= \frac{-RK\omega \pm \sqrt{(RK\omega)^2 + V_{\max}^2 - Z(K\omega)^2}}{Z} \text{ for } (v_d^*)^2 + (v_q^*)^2 \geq V_{\max}^2 \text{ and } (RK\omega)^2 + V_{\max}^2 - Z(K\omega)^2 > 0 \\ &= 0 \text{ otherwise.}\end{aligned}$$

with + for motoring and − braking.

7.2 Case 2

Operating under only the voltage constraint, the optimal field weakening current is given by

$$i_d^* = -\frac{(n_p \omega L)(K\omega)}{Z}$$

and the corresponding quadrature current is

$$i_q^* = \frac{\pm V_{max}\sqrt{Z} - K\omega R}{Z}$$

This is the overall optimum as long as

$$(i_d^*)^2 + (i_q^*)^2 \leq I_{\max}^2$$

If this constraint is violated, for $-I_{\max} < i_d^* < 0$, then the quadrature current is limited to

$$i_q^* = \pm\sqrt{I_{\max}^2 - (i_d^*)^2}$$

otherwise $i_q^* = 0$. In summary,with $i_d^* = -(n_p\omega L)(K\omega)/Z$ the direct current reference, the corresponding quadrature current is

$$\begin{aligned} i_q^* &= \frac{\pm V_{max}\sqrt{Z} - K\omega R}{Z} \text{ for } (i_d^*)^2 + (i_q^*)^2 \leq I_{\max}^2 \\ &= \pm\sqrt{I_{\max}^2 - (i_d^*)^2} \text{ for } (i_d^*)^2 + (i_q^*)^2 \geq I_{\max}^2 \text{ and } -I_{\max} < i_d^* < 0 \\ &= 0 \text{ otherwise} \end{aligned}$$

with + for motoring and − braking.

7.3 Case 3

This case gives a direct current reference as $i_d^* = -\sqrt{I_{max}^2 - (i_q^*)^2}$ and the corresponding quadrature current is

$$\begin{aligned} i_q^* &= \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \text{ for } \beta^2 - 4\alpha\gamma \geq 0 \\ &= 0 \text{ otherwise} \end{aligned}$$

with + for motoring and − braking.

8 Appendix A: Parameters - Back EMF and Torque Constant

The manufacturer's parameters (Aerotech, Inc., Pgh PA) are given as follows:

Line to line inductance = .028 H

Line to line Resistance = .5 Ohms

Line to Line Back EMF constant = $K_b^{\ell-\ell} = 23.6\ \frac{\text{Volts(peak)}}{\text{Krpm}}$

Torque Constant = $K_T = 0.28\ \frac{\text{N-m}}{\text{Amp(RMS)}}$

The line to line inductance and resistance measurements are done with the rotor held fixed and applying voltage between phase 1 and 2 with phase 3 open circuited. Subtracting the second equation from the first in 3, with $i_{S2} = -i_{S1}, \omega = 0$ gives

$$2(L_S + M)\frac{di_{S1}}{dt} = v_{S1} - v_{S2} - 2Ri_{S1}$$

Consequently, $2(L_S + M) = .028$ H, or, using $M = L_S/2$, gives

$$\begin{aligned} L_S &= \frac{.028}{3} \text{ H} \\ R &= \frac{.5}{2} \text{ Ohms} \end{aligned}$$

The line to line back emf voltage is measured by externally driving the motor at constant speed with the phases open circuited and measuring the voltage between two phases. In this setup, the voltages are given by

$$
\begin{aligned}
v_{S1} &= -K_m\omega\sin(n_p\theta) \\
v_{S2} &= -K_m\omega\sin(n_p\theta - \frac{2\pi}{3}) = -K_m\omega\left(-.5\sin(n_p\theta) - \sqrt{3}/2\cos(n_p\theta)\right) \\
v_{S3} &= -K_m\omega\sin(n_p\theta - \frac{4\pi}{3}) = -K_m\omega\left(-.5\sin(n_p\theta) + \sqrt{3}/2\cos(n_p\theta)\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
v_{S2} - v_{S3} &= \sqrt{3}K_m\omega\cos(n_p\theta) \\
\frac{(v_{S2} - v_{S3})_{peak}}{\omega} &= \sqrt{3}K_m\,\frac{\text{Volts}}{\text{rads/sec}}
\end{aligned}
$$

$$
\begin{aligned}
K_b^{\ell-\ell} &= \sqrt{3}K_m\frac{2\pi/60\text{ rads/sec}}{\text{rpm}}\frac{1000\text{ rpm}}{\text{Krpm}} \\
K &= \sqrt{\frac{3}{2}}K_m = \frac{1}{\sqrt{2}}K_b^{\ell-\ell}\frac{\text{Volts(peak)}}{\text{Krpm}}\frac{\text{Krpm}}{1000\text{rpm}}\frac{\text{rpm}}{2\pi/60\text{rads/sec}} \\
&= \frac{1}{\sqrt{2}}23.6\frac{1}{1000}\frac{60}{2\pi} = 0.159\ \frac{\text{Volts(peak)}}{\text{rads/sec}}
\end{aligned}
\tag{61}
$$

In steady-state where

$$
\begin{aligned}
i_{S1} &= I_{3ph}\cos(\omega_e t) \\
i_{S2} &= I_{3ph}\cos(\omega_e t - \frac{2\pi}{3}) \\
i_{S3} &= I_{3ph}\cos(\omega_e t - \frac{4\pi}{3})
\end{aligned}
$$

so that using the 3-2 phase transformation results in

$$
\begin{bmatrix} i_a \\ i_b \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{3}{2}}I_{3ph}\cos(\omega_e t) \\ \sqrt{\frac{3}{2}}I_{3ph}\sin(\omega_e t) \end{bmatrix}
$$

$$
\begin{aligned}
\tau &= -\sqrt{\frac{3}{2}}K_m i_a\sin(n_p\theta) + \sqrt{\frac{3}{2}}K_m i_b\cos(n_p\theta) \\
&= \sqrt{\frac{3}{2}}K_m\sqrt{\frac{3}{2}}I_{3ph}\sin(\omega_e t - n_p\theta) \\
&= \sqrt{2}\frac{3}{2}K_m I_{3ph\_rms}\sin(\omega_e t - n_p\theta)
\end{aligned}
$$

The manufacturer's value for the torque constant K_T is the torque per unit of RMS phase current which this expression shows is equal to $K_T = \sqrt{2}\frac{3}{2}K_m\ \frac{\text{N-m}}{\text{Amp(rms)}}$.

$$
K_m = \frac{1}{\sqrt{2}}\frac{2}{3}K_T
$$

where K_T is given in units of $\frac{\text{N-m}}{\text{Amp(peak)}}$ while K_m is in $\frac{\text{N-m}}{\text{Amp(rms)}}$. Finally,

$$K = \sqrt{\frac{3}{2}} K_m = \frac{K_T}{\sqrt{3}} = \frac{0.28}{\sqrt{3}} \frac{\text{N-m}}{\text{Amp(peak)}} = 0.162 \frac{\text{N-m}}{\text{Amp(peak)}} \tag{62}$$

Consequently, computing K using the backemf constant (61) gives the same value as computing K using the torque constant (62) as they must.