Root locus rules for polynomials with complex coefficients*

Arnau Dòria-Cerezo[1] and Marc Bodson[2]

***Abstract*— Applications were found recently where the analysis of dynamic systems with a special structure could be simplified considerably by transforming them into equivalent systems having complex coefficients and half the number of poles. The design of controllers for such systems can be simplified in the complex representation, but requires techniques suitable for systems with complex coefficients. In the paper, the extension of the classical root locus method to systems with complex coefficients is presented. The results are applied with some advantages to a three-phase controlled rectifier.**

I. INTRODUCTION

The theory of linear control system design concerns almost exclusively systems with real coefficients. Since physical systems are described by state-space models or transfer functions with real parameters, it would not appear useful to relax this assumption. However, applications were found recently [5][10] where the analysis of electric machines could be simplified considerably by transforming them into equivalent systems with complex coefficients. The applications included a self-excited induction generator and a doubly-fed induction motor/generator with active/reactive power control. For the transformation to apply, in general, the systems need to satisfy symmetry conditions that enable a reduction of the order of the system by a factor of 2. In [5][10], the extension of the classical Hurwitz test to systems with complex coefficients [13] yielded the derivation of analytic stability conditions that could not be obtained for the original (real) system, due to its higher dimension.

Other examples of dynamic systems having transfer functions with complex coefficients are somewhat rare, but can be found in asymmetric bandpass and band-rejection filters [2], mobile radio communication filtering algorithms [15], whirling shafts [8], and some mechanical systems [14]. Control theory tools for systems with complex coefficients are also very limited. The Hurwitz and Routh-Hurwitz tests for complex polynomials were applied in [5][10] (see also [1][3][6][9]). Later on, extended versions of Kharitonov's criterion for polynomials with uncertain complex coefficients were studied in [4][7][8][16][17][18].

In this paper, we consider the design of controllers for systems with complex coefficients using the root locus method. The root locus method was developed by W.R. Evans in the 40's [11][12], and is a fundamental tool that is taught in first courses on feedback systems. The root locus method for the general case with complex polynomials presents both interesting similarities and peculiar differences compared to the conventional root locus. Curiously, the method enables the design of control algorithms with properties that have no equivalent in the classical root locus design. The results are illustrated at the end of the paper with the example of a three-phase controlled rectifier.

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[1]A. Dòria-Cerezo is with the Department of Electrical Engineering and the Institute of Industrial and Control Engineering, Universitat Politècnica de Catalunya, 08028 Barcelona, Spain arnau.doria@upc.edu

[2]M. Bodson is with the Department of Electrical and Computer Engineering, University of Utah, Salt Lake City, UT 84112, USA bodson@eng.utah.edu

II. PROBLEM STATEMENT

In this paper, we consider systems with the standard state-space model

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

where the state, input, and output vectors can be split into two vectors of equal dimensions such that

$$\begin{aligned} x &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad u = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ y &= \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \end{aligned} \tag{2}$$

while

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & -A_{21} \\ A_{21} & A_{11} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & -B_{21} \\ B_{21} & B_{11} \end{pmatrix} \\ C &= \begin{pmatrix} C_{11} & -C_{21} \\ C_{21} & C_{11} \end{pmatrix} \end{aligned} \tag{3}$$

For systems satisfying (3), the following facts apply.

Fact 1: The transfer function matrix from u to y can be partitioned similarly with

$$\begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} = \begin{pmatrix} H_{11}(s) & -H_{21}(s) \\ H_{21}(s) & H_{11}(s) \end{pmatrix} \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} \tag{4}$$

Proof: Considering the structure of the A matrix, partition the matrix inverse as

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} sI - A_{11} & A_{21} \\ -A_{21} & sI - A_{11} \end{pmatrix}^{-1} \tag{5}$$

Then

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} sI - A_{11} & A_{21} \\ -A_{21} & sI - A_{11} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{6}$$

so that

$$\begin{aligned} M_{11}(sI - A_{11}) - M_{12}A_{21} &= I \\ M_{11}A_{21} + M_{12}(sI - A_{11}) &= 0 \\ M_{21}(sI - A_{11}) - M_{22}A_{21} &= 0 \\ -M_{21}A_{21} + M_{22}(sI - A_{11}) &= I \end{aligned} \tag{7}$$

The first and second equations of (7) give the values of M_{11} and M_{12}

$$\begin{aligned} M_{11} &= \left((sI - A_{11}) + A_{21} (sI - A_{11})^{-1} A_{21} \right)^{1} \\ M_{12} &= -M_{11} A_{21} (sI - A_{11})^{-1} \end{aligned} \tag{8}$$

while the third and fourth equations of (7) show that M_{22} and $-M_{21}$ satisfy equations identical to the first two equations defining M_{11} and M_{12}. Therefore

$$M_{22} = M_{11}, \quad M_{21} = -M_{12} \tag{9}$$

It is straightforward to show that the structure of the inverse matrix is preserved by pre- and post-multiplication by the C and B matrices. Therefore, the transfer function matrix (4) has the same structure. ■

Fact 2: Defining complex vectors

$$y_c = y_1 + jy_2, \quad u_c = u_1 + ju_2 \tag{10}$$

the signal y_c is the output of a system with input u_c and transfer function

$$H_c(s) = H_{11}(s) + jH_{21}(s) \tag{11}$$

where $H_{11}(s)$ and $H_{21}(s)$ are the sub-matrices of (4). One also has that

$$H_c(s) = C_c(sI - A_c)^{-1} B_c \tag{12}$$

where

$$A_c = A_{11} + jA_{21}, \quad B_c = B_{11} + jB_{21}, \quad C_c = C_{11} + jC_{21} \tag{13}$$

Proof: We have

$$\begin{aligned} y_c(s) &= H_{11}(s)u_1(s) - H_{21}(s)u_2(s) \\ &\quad + j\left(H_{21}(s)u_1(s) + H_{11}(s)u_2(s)\right) \\ &= \left(H_{11}(s) + jH_{21}(s)\right)\left(u_1(s) + ju_2(s)\right) \end{aligned} \tag{14}$$

Similarly,

$$\begin{aligned} \dot{x}_1 + j\dot{x}_2 &= (A_{11} + jA_{21})(x_1 + jx_2) \\ &\quad + (B_{11} + jB_{21})(u_1 + ju_2) \\ y_c &= y_1 + jy_2 = (C_{11} + jC_{21})(x_1 + jx_2) \end{aligned} \tag{15}$$

so that the results follow. ■

According to (11)

$$H_c(s) = \frac{N_{OL}(s)}{D_{OL}(s)} \tag{16}$$

where $N_{OL}(s)$ and $D_{OL}(s)$ are polynomials in s. (11) suggests that $D_{OL}(s)$ has real coefficients, while $N_{OL}(s)$ has complex coefficients. However, (12) indicates that the number of poles of $H_c(s)$ must be $1/2$ the number of poles of the original system. Thus, half the poles must be cancelled by zeros in (11). For a system with complex poles, both the numerator and denominator of $H_c(s)$ have complex coefficients.

Fact 3: Any root of $\det(sI - A_c) = 0$ is a root of $\det(sI - A) = 0$. On the other hand, if s_0 is a root of $\det(sI - A) = 0$, then either s_0 or its complex conjugate s_0^* is a root of $\det(sI - A_c) = 0$.

The fact is proved in [5]. First, it implies specific properties of the polynomial $\det(sI - A)$. Specifically, due to the special structure of (3), the roots of $\det(sI - A) = 0$ must be either complex pairs or double real pairs. In other words, there cannot be single real roots. Next, each root of $\det(sI - A_c) = 0$ is one of the roots in a pair of roots of $\det(sI - A) = 0$. Thus, knowledge of the eigenvalues of A_c implies knowledge of the eigenvalues of the original matrix A: all the poles of the original system can be obtained from the roots of $\det(sI - A_c)$.

Corollary 1: Consider a system with state-space model (3) and (4). Assume that output feedback is applied with

$$u(t) = -k \begin{pmatrix} k_{11} & -k_{21} \\ k_{21} & k_{11} \end{pmatrix} y(t) \tag{17}$$

where k is a real, adjustable gain, and k_{11}, k_{21} are real, fixed parameters. Then, the closed-loop poles are given by the roots of

$$D_{OL}(s) + kk_C N_{OL}(s) = 0 \tag{18}$$

and their complex conjugates, where

$$k_C = k_{11} + jk_{21} \tag{19}$$

Example. Consider a system with

$$A = 0, \quad B = I, \quad C = I \tag{20}$$

where I is the identity matrix. Let $k_{11} = \cos(\alpha)$ and $k_{21} = \sin(\alpha)$ for some angle α. The closed-loop system is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -k\cos(\alpha) & k\sin(\alpha) \\ -k\sin(\alpha) & -k\cos(\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{21}$$

which corresponds to the complex system

$$\dot{x}_c = -kk_C x_c. \tag{22}$$

where $k_C = e^{j\alpha}$. The complex system has complex parameters, but half the number of states. For varying k, the locus of the poles is composed of the single branch

$$s = ke^{j\alpha} \tag{23}$$

The poles of the original system are given by

$$s_1 = ke^{j\alpha}, \quad s_2 = ke^{-j\alpha} \tag{24}$$

Note that the angle in the complex plane is α and is not constrained by the usual root locus rules (which would require $\alpha = 0$). This is possible because the root locus for the original system is determined by the characteristic equation

$$\det \begin{pmatrix} s + k\cos(\alpha) & -k\sin(\alpha) \\ k\sin(\alpha) & s + k\cos(\alpha) \end{pmatrix} = s^2 + 2k\cos(\alpha)s + k^2 \tag{25}$$

Even though the characteristic polynomial is linear in k in the complex domain, the dependency in the real domain is polynomial. The root locus for the real system cannot be drawn using conventional root locus rules. The feedback law (17) is a *multivariable* feedback law, as opposed to the single-input single-output feedback law assumed in the conventional root locus. ◇

III. ROOT LOCUS WITH COMPLEX COEFFICIENTS

Consider the polynomial

$$D_{CL}(s) = D_{OL}(s) + kk_C N_{OL}(s) \tag{26}$$

where $N_{OL}(s)$ and $D_{OL}(s)$ are polynomials with complex coefficients. The degrees of $N_{OL}(s)$ and $D_{OL}(s)$ are m and n, respectively, with $q = n - m$ and $q > 0$ (we assume that the transfer function is strictly proper). The coefficients with the highest powers of s are equal to 1 in both polynomials. k is real and k_c is complex. By definition, the root locus is the locus of the roots of $D_{CL}(s) = 0$ as k varies from 0 to infinity.

In general, the polynomials $D_{OL}(s)$ and $N_{OL}(s)$ can be factored as

$$\begin{aligned} D_{OL}(s) &= (s-p_1)(s-p_2)\dots(s-p_n) \\ N_{OL}(s) &= (s-z_1)(s-z_2)\dots(s-z_m) \end{aligned} \tag{27}$$

where $p_1, p_2, \dots, p_n$ and $z_1, z_2, \dots, z_m$ are the roots of $D_{OL}(s)$ and $N_{OL}(s)$. Compared to the polynomial form

$$\begin{aligned} D_{OL}(s) &= s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n \\ N_{OL}(s) &= s^m + b_1 s^{m-1} + \dots + b_{m-1}s + b_m \end{aligned} \tag{28}$$

one has

$$a_1 = -\sum_{i=1}^{n} p_i, \quad b_1 = -\sum_{i=1}^{m} z_i. \tag{29}$$

From the characteristic equation

$$D_{OL}(s) + kk_C N_{OL}(s) = 0 \tag{30}$$

a magnitude condition can be derived (assuming[1] $k > 0$)

$$k|k_C| \left| \frac{N_{OL}(s)}{D_{OL}(s)} \right| = 1 \tag{31}$$

as well as a phase condition

$$\angle k_c + \angle \left(\frac{N_{OL}(s)}{D_{OL}(s)} \right) = \pm r\pi \tag{32}$$

where $r = 1, 3, 5...$ The phase condition can also be written as

$$\angle k_C + \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = \pm r\pi. \tag{33}$$

Since q, $k > 0$, the order of the polynomial $D_{CL}(s)$ is always n. Hence, it must always have exactly n roots, and the first rule can be derived as in the real case.

Rule 1: **Number of branches of the root locus.** The number of branches of the root locus is equal to the degree of the characteristic polynomial, n.

A property that does not follow from the real case is that the branches of the complex root-locus do not have to be symmetric with respect to the real axis. There is also no guarantee that any portion of the real axis belongs to the root-locus.

[1] An alternate set of rules can be derived for $k < 0$, often referred as Negative Root Locus.

The second and third rules define the starting points (the roots of $D_{CL}(s) = 0$ for $k = 0$) and the end points (the roots of $D_{CL}(s) = 0$ as k tends to infinity) of the root locus.

Rule 2: **Starting points of the root locus.** The root locus of $D_{CL}(s)$ starts at the open-loop poles ($D_{OL}(s) = 0$).

Proof: Replacing $k = 0$ in (26), with $k_C \neq 0$, it automatically follows that, $D_{CL}(s) = D_{OL}(s) = 0$. ■

Rule 3: **End points of the root locus.** m of the branches of the root-locus converge to the roots of $N_{OL}(s)$, while the other $n - m$ roots converge to infinity along asymptotes whose angles with respect to the real axis are defined by the $n - m$ complex roots of

$$s_i = \sqrt[n-m]{-k_C}, \qquad i = 1, \dots, n-m \tag{34}$$

The center of the asymptotes is located at

$$c = \frac{1}{n-m} \left(\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i \right) \tag{35}$$

where p_i and z_i are the roots of $D_{OL}(s)$ and $N_{OL}(s)$, respectively.

Note that, in contrast to the case with a polynomial with real coefficients, the center of the asymptotes can be everywhere of the complex plane. Also, the asymptotes do not have to be symmetric with respect to the real axis.

Proof: Assuming $|k_C| \neq 0$, if $k \to \infty$, the magnitude condition (31) holds only if

- $N_{OL}(s) \to 0$, or
- $s \to \infty$.

The first condition directly implies that m roots of $D_{CL}(s)$ tend to the roots of $N_{OL}(s)$ as $k \to \infty$.

Supposing $N_{OL}(s) \neq 0$, the characteristic equation (30) can be written as

$$-kk_C = \frac{D_{OL}(s)}{N_{OL}(s)} \tag{36}$$

With polynomial division

$$\begin{aligned} -kk_C = s^{n-m} \Bigg(1 + \frac{a_1 - b_1}{s} \\ + \frac{a_2 - b_2 - b_1(a_1 - b_1)}{s^2} + \dots \Bigg). \end{aligned} \tag{37}$$

As $s \to \infty$, the condition becomes

$$-kk_C = s^{n-m} \left(1 + \frac{a_1 - b_1}{s} \right). \tag{38}$$

Then, we may write

$$k^{\frac{1}{n-m}} (-k_C)^{\frac{1}{n-m}} = s \left(1 + \frac{a_1 - b_1}{s} \right)^{\frac{1}{n-m}}, \tag{39}$$

and keeping only the first-order term of the Taylor series, and solving for s gives

$$s = k^{\frac{1}{n-m}} (-k_C)^{\frac{1}{n-m}} - \frac{a_1 - b_1}{n-m}. \tag{40}$$

where $r = 1, 3, 5 \ldots$ Hence, as $k \to \infty$, $n - m$ roots of (30) go to infinity along the asymptotes radiating from (35) with an angle with respect to the real axis given by (34). ∎

If two roots approach each other as k varies, a break-away point can occur. In contrast to the classical root locus, break-away points can appear in any point of the complex plane (see the example of Fig. 4 discussed later in the paper).

Rule 4: **Break-away and break-in points.** A break-away point s_0 must be a root of (30) that also fulfills

$$\frac{\left.\frac{\mathrm{d}D_{OL}(s)}{\mathrm{d}s}\right|_{s=s_0}}{\left.\frac{\mathrm{d}N_{OL}(s)}{\mathrm{d}s}\right|_{s=s_0}} = \frac{D_{OL}(s)|_{s=s_0}}{N_{OL}(s)|_{s=s_0}}. \tag{41}$$

Proof: A polynomial $D_{CL}(s)$ has more than one root at $s = s_0$ if and only if

$$D_{CL}(s_0) = 0 \tag{42}$$

and

$$\left.\frac{\mathrm{d}D_{CL}(s)}{\mathrm{d}s}\right|_{s=s_0} = 0. \tag{43}$$

Then, using (30) in (42) and (43), condition (41) is obtained. ∎

Rule 5: **Angle of departure from complex poles.** The angle of departure from a complex pole, p_j, of $D_{OL}(s)$ is given by

$$\theta_j^d = \pi + \angle k_C + \sum_{i=1}^{m} \angle(p_j - z_i) - \sum_{i=1, i\neq j}^{n} \angle(p_j - p_i). \tag{44}$$

Proof: From the phase condition (33), the angle at p_j is given by

$$\angle(s - p_j) = \pm r\pi + \angle k_C + \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1, i\neq j}^{n} \angle(s - p_i), \tag{45}$$

The angle of departure is defined as $s \to p_j$, which yields (44). ∎

Rule 6: **Angle of arrival at complex zeros.** The angle of arrival at a complex zero, z_j, of $N_{OL}(s)$ is given by

$$\theta_j^a = \pi - \angle k_C - \sum_{i=1, i\neq j}^{m} \angle(z_j - z_i) + \sum_{i=1}^{n} \angle(z_j - p_i). \tag{46}$$

Proof: From the phase condition (33), the angle at z_j is given by

$$\angle(s - z_j) = \pm r\pi - \angle k_C - \sum_{i=1, i\neq j}^{m} \angle(s - z_i) + \sum_{i=1}^{n} \angle(s - p_i), \tag{47}$$

The angle of arrival is defined as $s \to z_j$, which yields (46). ∎

Rule 7: **Imaginary axis crossing.** The intersection of the root locus with the imaginary axis can be found by separating the equation

$$D_{OL}(j\omega) + k k_C N_{OL}(j\omega) = 0 \tag{48}$$

into real and imaginary parts and finding values of k for which real solutions exist for ω. Alternatively, the values for k can be obtained from the complex Hurwitz test (see Theorem 1) given below.

Theorem 1: The polynomial $P(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \ldots + \alpha_n$, where $\alpha_k = a_k + j b_k$ and $k = 1, 2, \ldots, n$, has all its zeros in the half-plane $\mathbb{R}(s) < 0$ if and only if the determinants, $\Delta_1 \ldots \Delta_k$,

$$\Delta_1 = a_1 \tag{49}$$

and

$$\Delta_k = \begin{vmatrix} a_1 & a_3 & a_5 & \ldots & a_{2k-1} & -b_2 & -b_4 & \ldots & -b_{2k-2} \\ 1 & a_2 & a_4 & \ldots & a_{2k-2} & -b_1 & -b_3 & \ldots & -b_{2k-3} \\ \vdots & & & \ddots & \vdots & & & \ddots & \vdots \\ 0 & & & \ldots & a_k & 0 & & \ldots & -b_{k-1} \\ 0 & b_2 & b_4 & \ldots & b_{2k-2} & a_1 & a_3 & \ldots & a_{2k-3} \\ 0 & b_1 & b_3 & \ldots & b_{2k-3} & 1 & a_2 & \ldots & a_{2k-4} \\ \vdots & & & \ddots & \vdots & & & \ddots & \vdots \\ 0 & & & \ldots & b_k & 0 & & \ldots & a_{k-1} \end{vmatrix} \tag{50}$$

for $k = 2, 3, \ldots, n$ and $a_r = b_r = 0$ for $r > n$, are all positive.

Proof: See Theorem 3.2 of [13]. ∎

IV. Application: A Three-Phase Controlled Rectifier

Power systems turn out to offer significant opportunities for applying the root locus method to systems with complex coefficients. As a first, illustrative and relatively simple example, we apply the complex root locus rules to the current control of a three-phase rectifier in the dq reference frame. Figure 1 shows the circuit scheme for one axis (both the d and the q axes satisfy the same circuit). L is the filter inductance, r represents the inductance losses, v is the grid voltage, and e is the voltage generated by the switch positions of the full bridge inverter.

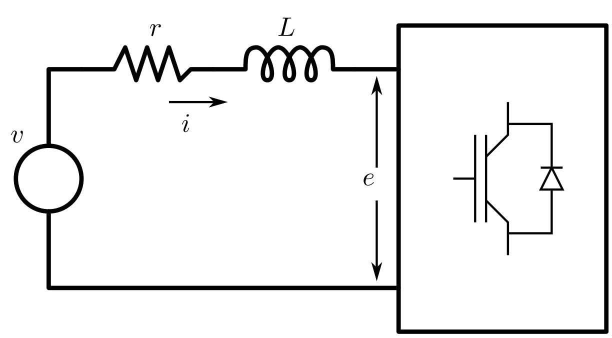

Fig. 1. Simplified circuit scheme of a three-phase rectifier.

Assuming a balanced grid, the RL filter of the rectifier has the following linear dynamics

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = -(r\boldsymbol{I} + \omega_s L \boldsymbol{J})i - e + v \tag{51}$$

where $i = \mathrm{col}(i_d, i_q) \in \mathbb{R}^2$ are the dq-currents, $v = \mathrm{col}(v_d, v_q) \in \mathbb{R}^2$ are the dq-grid voltages, $e = \mathrm{col}(e_d, e_q) \in \mathbb{R}^2$, are the averaged voltages given by the switching policy, L and r are the RL filter parameters, ω_s is the grid frequency and

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{52}$$

A typical control scheme for this problem has a feedback compensator that decouples the dq-currents and translates the multi-input multi-output control problem into two single-input single-output control designs. The complex root-locus suggests an interesting approach with different properties. We consider a PI-controller

$$e = -k_P(i^* - i) - \frac{k_P}{T_i}\int (i^* - i)\mathrm{d}t, \tag{53}$$

where k_p is the proportional gain, T_i is the integral time value and the upper index $(\cdot)^*$ refers to the desired current values. Applying the Laplace transform in (51) and (53) the closed-loop system becomes

$$\begin{pmatrix} Ls + r\boldsymbol{I} + \omega_s L\boldsymbol{J} & \boldsymbol{I} \\ -k_P(s + \frac{1}{T_i})\boldsymbol{I} & s\boldsymbol{I} \end{pmatrix} \begin{pmatrix} I(s) \\ E(s) \end{pmatrix} = \begin{pmatrix} V(s) \\ -k_P(s + \frac{1}{T_i})I^*(s) \end{pmatrix} \tag{54}$$

whose stability is given by the determinant of the matrix on the left-hand side, *i.e.,*

$$\begin{aligned} D(s) \;=\; & L^2 s^4 + 2L(r + k_P)s^3 \\ & + \left((r + k_P)^2 + 2L\frac{k_P}{T_i} + \omega_s^2 L^2\right)s^2 \\ & + 2\frac{k_P}{T_i}(k_P + r)s + \frac{k_P^2}{T_i^2}. \end{aligned} \tag{55}$$

The analysis of the poles of (55) is complicated [2], especially if we are interested in determining the influence of the parameters. Moreover, rewriting (55)

$$\begin{aligned} D(s) \;=\; & \left(s^2 + \frac{2}{T_i}s + \frac{1}{T_i^2}\right)k_P^2 \\ & + 2s\left(s^2 L + \left(r + \frac{L}{T_i}\right)s + \frac{r}{T_i}\right)k_P \\ & + L^2 s^4 + 2rLs^3 + (\omega_s^2 L^2 + r^2)s^2 \end{aligned} \tag{56}$$

shows that it is not possible to apply the conventional root-locus rules, because the polynomial is not linear in k_P. These facts motivate the use of polynomials with complex coefficients.

A. Generalized PI controller

The original PI controller (53) did not consider cross-terms between the dq axes. However, the use of the complex representation not only allows to go further in the analysis, but also to consider a more general PI structure

$$e = -\boldsymbol{K}_P(i^* - i) - \boldsymbol{K}_I\int (i^* - i)\mathrm{d}t, \tag{57}$$

where

$$\begin{aligned} \boldsymbol{K}_P &= k_{PR}\boldsymbol{I} + k_{PI}\boldsymbol{J} \\ \boldsymbol{K}_I &= k_{IR}\boldsymbol{I} + k_{II}\boldsymbol{J}. \end{aligned} \tag{58}$$

Alternatively, the $\boldsymbol{K}_P$ and $\boldsymbol{K}_I$ gains can be defined as

$$\begin{aligned} \boldsymbol{K}_P &= k_P(\boldsymbol{I} + \delta_p\boldsymbol{J}) \\ \boldsymbol{K}_I &= \frac{k_P}{T_i}(\boldsymbol{I} + \delta_i\boldsymbol{J}) \end{aligned} \tag{59}$$

where $k_P := k_{PR}$, $\delta_p := \frac{k_{PI}}{k_{PR}}$, $T_i := \frac{k_{PR}}{k_{IR}}$, $\delta_i := \frac{k_{II}}{k_{IR}}$, and $k_P, \delta_p, T_i, \delta_i > 0$.

Applying the Laplace transform in (51) and (57) the closed-loop system becomes

$$\boldsymbol{A}(s)\begin{pmatrix} I(s) \\ E(s) \end{pmatrix} = \begin{pmatrix} V(s) \\ -(\boldsymbol{K}_P s + \boldsymbol{K}_I)I^*(s) \end{pmatrix} \tag{60}$$

where

$$\boldsymbol{A}(s) = \begin{pmatrix} (Ls + r)\boldsymbol{I} + \omega_s L\boldsymbol{J} & \boldsymbol{I} \\ -(\boldsymbol{K}_P s + \boldsymbol{K}_I) & s\boldsymbol{I} \end{pmatrix}, \tag{61}$$

and defining $\mathcal{I}(s) = I_d(s) + jI_q(s)$, $\mathcal{V}(s) = V_d(s) + jV_q(s)$, $\mathcal{E}(s) = E_d(s) + jE_q(s)$, (60) can be rewritten as

$$\boldsymbol{\mathcal{A}}(s)\begin{pmatrix} \mathcal{I}(s) \\ \mathcal{E}(s) \end{pmatrix} = \begin{pmatrix} \mathcal{V}(s) \\ -(K_P s + K_I)\mathcal{I}^*(s) \end{pmatrix}, \tag{62}$$

with

$$\boldsymbol{\mathcal{A}}(s) = \begin{pmatrix} Ls + r + j\omega_s L & 1 \\ -(K_P s + K_I) & s \end{pmatrix} \tag{63}$$

and

$$\begin{aligned} K_P &= k_P(1 + j\delta_p) \\ K_I &= \frac{k_P}{T_i}(1 + j\delta_i). \end{aligned} \tag{64}$$

B. Stability analysis

The stability of the closed-loop system (51)-(57) can be analyzed using the complex Hurwitz test. The complex polynomial $\det \boldsymbol{A}(s)$ has the form

$$D_{CL}(s) = Ls^2 + \big(r + k_P + j(\omega_s L + k_P\delta_p)\big)s + \frac{k_P}{T_i}(1 + j\delta_i), \tag{65}$$

and the stability conditions given by the complex Hurwitz test [13], with $L > 0$, reduce to

$$\begin{aligned} 0 &< r + k_P \\ 0 &< (r + k_P)^2 - L\frac{k_P}{T_i}\delta_i^2 + (r + k_P)(\omega_s L + k_P\delta_p)\delta_i. \end{aligned} \tag{66}$$

[2]Notice that if a decoupling term is included in (53), which is equivalent to cancel ω_s in (55), the polynomial $D(s)$ simplifies in

$$D(s) = \left(s(Ls + r) + k_P\left(s + \frac{1}{T_i}\right)\right)^2,$$

and makes the analysis more tractable.

C. Root locus analysis

For the root locus analysis, we consider that $\delta_i = \delta_p = \delta > 0$,

$$D_{CL}(s) = Ls^2 + \big(r + k_P + j(\omega_s L + k_P\delta)\big)s + \frac{k_P}{T_i}(1 + j\delta). \quad (67)$$

Then, the above complex polynomial can be written as (26) with

$$\begin{aligned} D_{OL}(s) &= Ls^2 + \big(r + j\omega_s L\big)s \\ N_{OL}(s) &= s + \frac{1}{T_i} \\ k &= k_P \\ k_C &= 1 + j\delta, \end{aligned} \quad (68)$$

where $n = 2$ and $m = 1$. Following the root locus rules from Section III, we get:

- **Rule 1: Number of branches of the root locus.** From (68), the number of branches is $n = 2$.
- **Rule 2: Starting points of the root locus.** The starting points are the roots of (68),

$$\begin{aligned} p_1 &= 0 \\ p_2 &= -\frac{r}{L} - j\omega_s. \end{aligned} \quad (69)$$

- **Rule 3: End points of the root locus.** One root converges to the root of (68), that is

$$z_1 = -\frac{1}{T_i}, \quad (70)$$

as $k \to \infty$, and the other root converges to ∞, along an asymptote with an angle

$$\theta^{\infty} = \arctan(-\delta), \quad (71)$$

and a centroid

$$c = -\frac{r}{L} + \frac{1}{T_i} - j\omega_s. \quad (72)$$

- **Rule 4: Break-away and break-in points.** From (41), the following complex equation is obtained

$$T_i L s^2 + 2Ls + r + j\omega_s L = 0. \quad (73)$$

Consequently, break-away (or break-in) points exists for those values of r, L, ω_s and T_i that satisfy (73) together with (67).
- **Rule 5: Angles of departure from the complex poles.** The angles of departure from complex poles are

$$\theta_1^d = \pi + \arctan(\delta) - \arctan\left(\frac{\omega_s L}{r}\right) \quad (74)$$

and

$$\begin{aligned} \theta_2^d &= \pi + \arctan(\delta) + \arctan\left(\frac{\omega_s}{\frac{r}{L} - \frac{1}{T_i}}\right) \\ &\quad - \arctan\left(\frac{\omega_s L}{r}\right) \end{aligned} \quad (75)$$

- **Rule 6: Angle of arrival at the complex zero.** The angle of arrival at z_1 is

$$\theta_1^a = -\arctan(\delta) + \arctan\left(\frac{\omega_s}{\frac{r}{L} - \frac{1}{T_i}}\right). \quad (76)$$

- **Rule 7: Imaginary axis crossing.** From (48), the following two conditions are obtained

$$\begin{aligned} L\omega^2 + (\omega_s L + \delta k_P)\omega - \frac{k_P}{T_i} &= 0 \\ (k_P + r)\omega + \delta\frac{k_P}{T_i} &= 0 \end{aligned} \quad (77)$$

The conditions imply

$$k_P^2 + \alpha_1 k_P + \alpha_0 = 0, \quad (78)$$

where

$$\begin{aligned} \alpha_1 &= \frac{T_i(r(1+\delta^2) + (r + \delta\omega_s L)) - \delta^2 L}{T_i(1+\delta^2)} \\ \alpha_0 &= \frac{r(r + \delta\omega_s L)}{1+\delta^2}. \end{aligned} \quad (79)$$

Since $\alpha_0 > 0$, the positive real solutions of k_P exists if $\alpha_1 < 0$ and $\alpha_1^2 - 4\alpha_0 > 0$.

The results of the root locus rules are summarized in Fig. 2. Notice that two different scenarios are obtained depending on $\frac{r}{L} > \frac{1}{T_i}$ (the case shown in Fig. 2) or $\frac{r}{L} < \frac{1}{T_i}$. However, some of the rules does not allow to obtain analytic conclusions and it must be studied numerically.

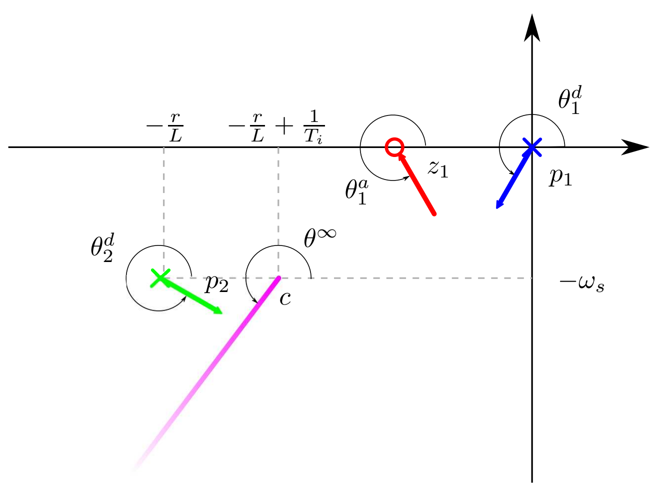

Fig. 2. Root-locus simplified scheme for $\frac{r}{L} > \frac{1}{T_i}$.

Replacing the following numerical values $r = 10$, $L = 1$, $\omega_s = 1$, $\delta = 10$ in (67) and (73), a break-in point appears for $T_i = T_i^{\mathrm{BK}} = 0.1651$, and is expected at $s = -5.4425 - j4.9254$ for $k_P = 0.8851$. From the conditions obtained in Rule 7, the root-locus crosses the imaginary axis for values $T_i < T_i^{\mathrm{IAC}}$, where $T_i^{\mathrm{IAC}} = 0.0761$. Fig. 3 shows the root-locus when varying T_i such that scenarios with different break-away points and imaginary axis crossings occur.

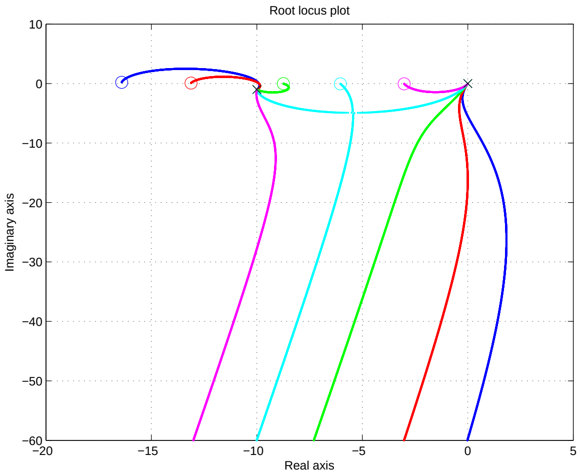

Fig. 3. Root locus of the 3-phase rectifier example. The T_i parameter takes the value: $T_i = 0.5T_i^{\text{IAC}}$ (blue line), $T_i = T_i^{\text{IAC}}$ (red line), $T_i = 1.5T_i^{\text{IAC}}$ (green line), $T_i = T_i^{\text{BK}}$ (cyan line) and$T_i = 2T_i^{\text{BK}}$ (magenta line).

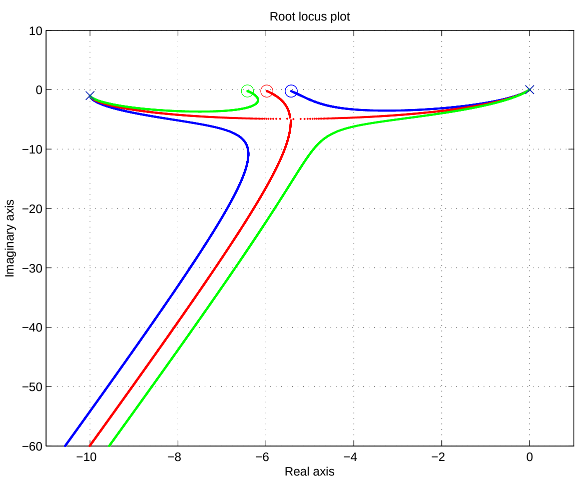

Fig. 4. Detail of the break-in point of the root locus. The T_i parameter takes the value: $T_i = 0.1538$ (green line), $T_i = T_i^{\text{BK}} = 0.1651$ (red line) and $T_i = 0.1818$ (blue line).

Overall, the complex root locus exhibits striking similarities with the real root locus. However, peculiar differences are also observed, namely:

- the root-locus (including the asymptotes) is not necessarily symmetric with respect to the real axis;
- in the case of a single asymptote, the angle with respect to the real axis can be set arbitrarily in the complex plane (this angle represents a degree of freedom not available in the conventional root locus);
- poles can merge and split off at arbitrary locations in the complex plane.

These differences in properties can be attributed to the fact that the complex root locus is, in general, associated with a multi-input multi-ouput feedback system. Even though the linearity of the complex characteristic polynomial in the variable gain is used, the characteristic polynomial of the original system is nonlinear in the gain parameter.

V. CONCLUSIONS

The classical root locus method was extended to polynomials with complex coefficients. The motivation for such an extension lied in the existence of systems which could be studied through equivalent complex systems of half the dimension or order. It was found that most but not all properties of the classical root locus extended with minor modifications to the complex case. Rules that did not extend yielded peculiar differences in characteristics reflective of a multivariable design.

The application of the results to a three-phase controlled rectifier was presented to illustrate the results. The complex representation simplifies the fourth order polynomial (55), which is nonlinear in k_P , into the second order polynomial (65) which is linear in k_P. In this manner, the problem can be solved analytically, in contrast with the traditional technique where only numerical results could be obtained. The complex notation has been used to describe various power systems, but not to perform design or stability analyses. The example shows that the root locus method for polynomials with complex coefficients can be a powerful tool for these purposes.

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