Final Exam

March 14, 4:30-6:30pm

Name: Solutions

(50 points total)

Selected Fourier Series Expansions

Square Wave: \[ f(t) = \frac{4}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n} \sin(n \cdot 2\pi ft) \]

Triangle Wave: \[ f(t) = \frac{8}{\pi^2} \sum_{n=1,3,5,...}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin(n \cdot 2\pi ft) \]

**Problem 1:** An integer-N PLL to be used for frequency synthesis is shown in Fig 1. The output frequency must range from 300 MHz to 400 MHz with 10 MHz channel spacings. Note that there is no charge pump, the output of the phase detector is a voltage which is filtered before being applied to the VCO. [13 points]

(a) What is an appropriate choice for the reference frequency \( f_{\text{ref}} \)? [1 point]

(b) What are the range of divide values \( N \) that will be used for this choice of reference frequency? [2 points]

(c) Derive expressions for the damping factor \( \zeta \) and the natural frequency \( \omega_n \) of the characteristic equation of the PLL. [6 points]

(d) Based on the constraints that there must be no peaking in the frequency response \( \zeta \geq 1/\sqrt{2} \) and that \( \omega_n \) must be minimized, choose a value for the product \( RC \). Assume that \( K_{PD} = 1 \) V/rad and \( K_{VCO} = 150,000 \) rad/sV. [4 points]
PROBLEM 1 (cont’d)

Figure 1: Integer-N PLL for Problem 1.

6) \( f_{\text{ref}} = 10 \, \text{MHz} \) (same as channel spacing)

b) \( N = 30 \) to \( 40 \) (300 MHz to 400 MHz)

c) \( H(s) = \frac{A}{1 + A \beta} = \frac{K_{PD} \cdot F(s) \cdot K_{VCO}}{s} \frac{1}{1 + K_{PD} \cdot K_{VCO} \cdot \frac{1}{N} \cdot F(s)} = \frac{1}{1 + sRC} \)

\[
= \frac{N \cdot K_{PD} \cdot K_{VCO}}{(1 + sRC) \cdot s \cdot N + K_{PD} \cdot K_{VCO}} \\
= \frac{N \cdot K_{PD} \cdot K_{VCO}}{s^2 \cdot R \cdot C \cdot N + s \cdot N + K_{PD} \cdot K_{VCO}}
\]

Characteristic equation: \( s^2 + 2\zeta \omega_n s + \omega_n^2 = s^2 + \frac{s}{R \cdot C \cdot N} + \frac{K_{PD} \cdot K_{VCO}}{R \cdot C \cdot N} \)

\[ \Rightarrow \omega_n = \sqrt{\frac{K_{PD} \cdot K_{VCO}}{R \cdot C \cdot N}} \checkmark, \quad \zeta = \frac{1}{2} \sqrt{\frac{R \cdot C \cdot N}{K_{PD} \cdot K_{VCO}}} \checkmark \]

\[ \zeta = \frac{1}{2} \sqrt{\frac{N}{R \cdot C \cdot K_{PD} \cdot K_{VCO}}} \checkmark \]
PROBLEM 1 (cont’d)

d) \( W_0 \) minimized \( \rightarrow \) make \( RC \) as large as possible

\[ \xi = \frac{1}{2} \sqrt{\frac{N}{RC K_p K_w}} \geq \frac{1}{\sqrt{2}} \]

\( \rightarrow \) worst for small value of \( N \)

\[ \frac{1}{2} \sqrt{\frac{350}{1.150 \times 10^6}} \geq \frac{1}{\sqrt{2}} \]

\[ \frac{1}{2} \sqrt{\frac{2 \times 10^{-4}}{RC}} \geq \frac{1}{\sqrt{2}} \]

\[ \frac{1}{\sqrt{2}} \frac{10^{-2}}{RC} \geq \frac{1}{\sqrt{2}} \]

\[ 10^{-4} \geq RC \]

\( \therefore RC \leq 10^{-4} \)

\( \rightarrow \) choose \( RC = 10^{-4} \)
Problem 2: An LC tank to be used in a VCO is shown in Fig. 2, where \( C = 1 \, \mu F \) and \( L = 1 \, \mu H \). \( R_C = 10^{-4} \, \Omega \) is the parasitic series resistance of the capacitor, and \( R_L = 10^{-4} \, \Omega \) is the parasitic series resistance of the inductor. [12 points]

(a) What is the resonant frequency of the tank? [1 point]
(b) What is the impedance seen looking into the tank at resonance? [5 points]
(c) What is the Q of the tank? [2 points]
(d) Suppose that a \(-g_m\) circuit is now going to be added to create a complete oscillator. Suggest an appropriate choice for the conductance \(-g_m\) to obtain start-up and sustained oscillation. Provide a sentence or two of explanation justifying your choice. [3 points]
(e) If a noise source within the VCO is creating white noise with a power of \( S_N(f) = N_o \), what will be the resulting noise power at the VCO output at an offset of \( \Delta \omega = 10^6 \) rad/s from the carrier? [2 points]

\[
\begin{align*}
\omega_o &= \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-6} \cdot 10^{-6}}} = 10^6 \text{ rad/s} \\
R_C &= \frac{1}{\sqrt{\left(\omega_o \cdot C\right)^2}} = \frac{1}{10^4 \left(10^{-6} \cdot 10^6\right)^2} = 10^4 \Omega \\
R_L &= \frac{\left(\omega_o \cdot L\right)^2}{R_C} = \left(10^{-6} \cdot 10^6\right)^2 = 10^4 \Omega \\
R_{in} &= 5 \times 10^3 \Omega
\end{align*}
\]
PROBLEM 2 (cont’d)

d) \[ Q = \frac{1}{\sqrt{1 - 1/c}} = \sqrt{\frac{5 \times 10^3}{10^{-3} / 10^{-3}}} = 5 \times 10^3 \]

d) Add conductances, need \[ -g_m + \frac{1}{R_p} \leq 0 \] \checkmark

\[ g_m \geq \frac{1}{R_p} \]

\[ g_m \geq \frac{1}{5 \times 10^3} = 2 \times 10^{-4} \]

\therefore \text{choose } g_m = 4 \times 10^{-4} \checkmark

\to \text{choosing double the min. value will ensure start-up.} \checkmark

e) Leeby’s equation: \[ \left| \frac{Y(\omega)}{X(\omega)} \right|^2 = \frac{1}{4 Q^2} \left( \frac{\omega_0}{\Delta \omega} \right)^2 \] \checkmark

\to \text{noise power } \Delta \omega = 10^6 \text{ rad/s} = \frac{N_0 \cdot 1}{4 \left(5 \times 10^3\right)^2 \left(\frac{10^6}{10^6}\right)^2} \]

\[ = \frac{N_0}{100 \cdot 10^6} \]

\[ = N_0 \cdot 10^{-8} \checkmark \]
PROBLEM 3: Consider the PLL shown in Fig. 3. The divider has been broken into two blocks, each dividing by $N$ for a total divide ratio of $N^2$. The first divider is noisy, injecting phase noise $\phi_N(t)$ as shown in the figure, and the second divider is ideal. [12 points]

(a) Calculate the noise transfer function that filters $\phi_N(t)$ before it appears at the output of the PLL. [5 points]

(b) Now assume that the loopfilter is a simple gain: $F(s) = K_{LPP}$. Sketch the noise transfer function derived in part (a), labelling any constant gain as well as the locations of any poles or zeros. [4 points]

(c) How would the gain of the transfer function you calculated change if the noise were injected by the second divider instead of the first? [2 points]

(d) Based on your answer to part (c), assuming you are using two different divider topologies in series and one is noisier than the other, in which order should they be placed? [1 point]

\[
\Phi_{\text{ref}} \rightarrow + \rightarrow K_{PD} \rightarrow F(s) \rightarrow K_{VCO/s} \rightarrow \Phi_{\text{out}}
\]

\[
\Phi_{\text{ref}} \rightarrow + \rightarrow \frac{1}{N} \rightarrow F(s) \rightarrow \frac{1}{N} \rightarrow \Phi_N
\]

\[
\Phi_{\text{out}}^{\cdot} = \frac{A}{1 + A\beta} = \frac{K_{PD} \cdot F(s) \cdot K_{VCO}}{1 + \frac{K_{PD} \cdot F(s) \cdot K_{VCO} \cdot N}{S \cdot N^2 + K_{PD} \cdot F(s) \cdot K_{VCO}}}
\]

Figure 3: PLL for Problem 3.
PROBLEM 3 (cont'd)

b) \[ \frac{\Phi_{out}}{\Phi_i} = \frac{\frac{K_{pp} K_{pf} K_{v_o} \cdot N}{S \cdot N^2 + K_{pp} K_{pf} K_{v_o}}}{\frac{1 + S \cdot N^2}{K_{pp} K_{pf} K_{v_o}}} = \frac{N}{1 + \frac{S \cdot N^2}{K_{pp} K_{pf} K_{v_o}}} \]

\[ 20 \log \left| \frac{\Phi_{out}}{\Phi_i} \right| = 20 \log (N) \]

\[ 20 \text{dB/dec} \]

\[ \log (\omega) \]

\[ \frac{K_{pp} K_{pf} K_{v_o}}{N^2} \]

c) The gain would increase to \( N^2 \), the pole remains the same.

d) The noisy divide should be placed first.
Problem 4: A PLL is being used as a frequency synthesizer to downconvert the received signal shown in Fig. 4 to an intermediate frequency of 1 MHz. You can assume that the only spectral components are those shown in the figure. The receiver is completely noiseless, but the designer has forgotten to include a band select filter at the receiver input. (Remember: the power of the downconverted signal will be proportional to the power of the original signal (denoted in the figure) and the power of the LO signal used for downconversion). [8 points]

(a) Assuming that the PLL output is an ideal squarewave, what will be the signal to noise ratio of the downconverted signal (hint: the Fourier series expansion for a square wave is given on the front page)? [4 points]

(b) Now assume that the PLL output is an ideal triangle wave, what will be the signal to noise ratio of the downconverted signal (hint: the Fourier series expansion for a triangle wave is given on the front page)? [3 points]

(c) If you are going to omit the band select filter on your receiver, would you prefer a local oscillator with a square wave or a triangle wave characteristic? [1 point]

![Figure 4: Frequency spectrum for Problem 4.](image)

\[ \text{Signal Power: } \left( \frac{4}{\pi} \right) \cdot 0.1 \text{ mW} \]

\[ \text{Noise Power: } \left( \frac{4}{\pi} \right) \left( \frac{1}{2} \right) \cdot 0.009 \text{ mW} \]

\[ \text{SNR} = \frac{0.1}{0.009} = 10 \log 10 \]

\[ = 10 \cdot \log (10) \]

\[ = 10 \text{ dB} \]
PROBLEM 4 (cont'd)

b) Signal Power: \( \left( \frac{g^2}{H^2} \right) \cdot 0.1 \approx 0 \)

Noise Power: \( \left( \frac{g^2}{n^2} \right) \cdot \left( \frac{1}{\pi^2} \right) \cdot 0.09 \)

\[ = \left( \frac{g^2}{n^2} \right) \cdot \frac{1}{81} \cdot 0.09 \]

\[ = 10 \cdot \log \left( \frac{0.1}{0.001} \right) \]

\[ = 10 \cdot \log (90) \]

\[ = 19.5 \text{ dB} \]

(c) We would prefer a triangle wave.
PROBLEM 5: A fractional-N PLL with spur cancellation is shown in Fig. 5(a). Here, the contents of the accumulator are used in conjunction with a current DAC to cancel the fractional phase error on each reference cycle to minimize the fractional spurs. An alternate spur cancellation architecture is shown in Fig. 5(b), where the same phase detector drives a second charge pump which injects an equal and opposite current to that of the original charge pump. Will this work? If so, compare its merits with that of the DAC method, and if not, explain why not. [3 points]

![Fractional-N PLLs for Problem 5.](image)

Figure 5: Fractional-N PLLs for Problem 5.

No, this will not work. In part (a) we are only cancelling the deterministic portion of the phase error, the random portion (i.e. phase noise) is still being suppressed by the PFD/CP. In (b) we are no longer cancelling the random part, and the feedback has been removed. For this reason, the PLL would also never lock.
PROBLEM 6: Write a short poem that expresses your feelings towards phase-locked loops, using the format of rhyming couplets. You can assume that poetic license has been granted. State any other assumptions. [2 points]

I love phase-locked loops,
They're better than shooting hoops.

EE 536 was my favourite class,
Even though I didn't pass.