ECE 6130 Rectangular Waveguides

Text Sections: 3.3

Chapter 3, Problem 3 (See Appendix I) and Derive the TM modes of a rectangular waveguide following the methods described here for TE modes.

<u>Rectangular Waveguides</u>

Recall: <u>Method of solution:</u>

1. Solve Helmholtz equation for either Hz (TE) or Ez (TM). This can be done analytically or numerically. In the analytical case, you guess the form of the solution, which will have several unknown constants (like magnitude, phase, number of cycles)

2. Use 3.19 to 3.23 to find transverse components from Ez or Hz.

3. Solve for the constants from the boundary conditions. In metal boundaries, these are that tangential E and normal H = 0 on the boundary. Now you have Ez or Hz.

4. Use Maxwell's equation to find the other E or H components.

TE Solution

1. Solve Helmholtz wave equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x, y) = 0$$

a) Use Method of Separation of Variables:
   $h_x(x,y) = X(x)\,Y(y)$
b) Substitute into wave equation (Helmholtz equation):

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + k_c^2 XY = 0$$

where $k_c^2 = k_x^2 + k_y^2$
c) Separate the Variables:

$$Y\left[\frac{d^2X}{dx^2} + k_x^2 X\right] = 0$$

$$X\left[\frac{d^2Y}{dy^2} + k_y^2 Y\right] = 0$$

d) "Guess" the form of the solution:

$h_z(x,y) = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y)$

The unknown constants are ABCD. Also k_x and k_y.

2. Solve for the unknown constants from boundary conditions.

a) Define the boundary conditions
Tangential E fields =0 on the metal surfaces (walls of the waveguide)
$e_x = 0$ at $y=0,b$
$e_y = 0$ at $x=0,a$

b) Obtain appropriate expressions for the boundary condition fields
From equations 3.19:
$Ex = (-j\omega\mu / k_c^2)\, \partial H_z / \partial y$
$Ey = (j\omega\mu / k_c^2)\, \partial H_z / \partial x$

So:
$e_x(x,y) = (-j\omega\mu / k_c^2)\, k_y\, (A \cos k_x x + B \sin k_x x)(- C \sin k_y y + D \cos k_y y)$
$e_y(x,y) = (j\omega\mu / k_c^2)\, k_x\, (-A \sin k_x x + B \cos k_x x)(C \cos k_y y + D \sin k_y y)$

c) Use boundary conditions to solve for ABCD:
Substitute $e_x = 0$ at $y=0,b$ into equations above.
$e_x(x,0) = (-j\omega\mu / k_c^2)\, k_y\, (A \cos k_x x + B \sin k_x x)(- C \sin k_y 0 + D \cos k_y 0)=0$
So, $D = 0$
Note that a "trivial solution" also exists if $k_y = 0$
$e_x(x,b) = (-j\omega\mu / k_c^2)\, k_y\, (A \cos k_x x + B \sin k_x x)(- C \sin k_y b + 0 \cos k_y b)=0$
when $k_y = n\pi/b$ and $k_x = m\pi/a$

Substitute $e_y = 0$ at $x=0,a$ into equations above:
$e_y(0,y) = (j\omega\mu / k_c^2)\, k_x\, (-A \sin k_x 0 + B \cos k_x 0)(C \cos k_y y + D \sin k_y y) = 0$
So, $B=0$
$e_y(x,a) = (j\omega\mu / k_c^2)\, k_x\, (-A \sin k_x a + B \cos k_x a)(C \cos k_y y + D \sin k_y y)=0$
So, $k_x = m\pi/a$

Now we can simplify the form:
$e_x(x,b) = (-j\omega\mu / k_c^2)(n\pi/b)(-AC) \cos(m\pi x/a) \sin(n\pi y/b)$
$e_x(x,b) = A_{mn} \cos(m\pi x/a) \sin(n\pi y/b)$

d) Apply constants to H:
$h_z(x,y) = (A \cos k_x x + 0 \sin k_x x)(C \cos k_y y + 0 \sin k_y y)$
$H_z(x,y) = AC \cos(m\pi x/a) \cos(n\pi y/b) = A_{mn} \cos(m\pi x/a) \cos(n\pi y/b)\, e^{-j\beta z}$

e) Ex, Ey, Hx,Hy components are found from equations 3.19 again.

TE_{mn} modes:

Modes are numbered m-n, indicating how many cosine wave cycles are in the waveguide.

Cutoff frequency for each mode is different:
$f_c = k_c / (2\pi\sqrt{\mu\varepsilon}) =$

$$f_{c_{mn}} = \frac{1}{2\pi\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

“Lowest Order Mode” Assuming b>a TE_{10} mode has the lowest cutoff frequency, so will be the first mode in the waveguide.