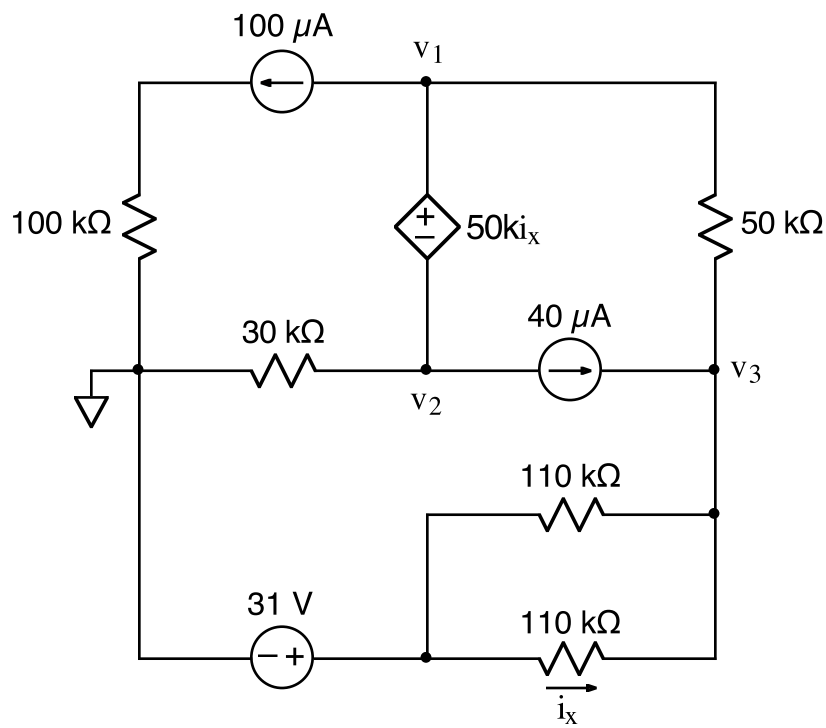
HOMEWORK #7 solution

Ex:



Use the node-voltage method to find v_1, v_2, and v_3.

SOL'N: Since we have a dependent source, we start by defining the dependent current in terms of the node voltages.

$$i_x = \frac{31 \text{ V} - v_3}{110 \text{ k}\Omega}$$

We have a supernode for nodes v_1 and v_2. We first sum all the currents out of a bubble drawn around nodes v_1 and v_2 with the dependent voltage source inside:

$$100 \ \mu\text{A} + \frac{v_1 - v_3}{50 \text{ k}\Omega} + \frac{v_2}{30 \text{ k}\Omega} + 40 \ \mu\text{A} = 0 \text{ A} \qquad (1)$$

We also write a voltage equation for the supernode:

$$v_1 - v_2 = 50 \text{ k}\Omega \cdot i_x = \frac{31 \text{ V} - v_3}{110 \text{ k}\Omega} \qquad (2)$$

For the v_3 node we have a conventional current sum:

$$\frac{v_3 - v_1}{50\ \text{k}\Omega} - 40\ \mu\text{A} + \frac{v_3 - 31\ \text{V}}{55\ \text{k}\Omega} = 0\ \text{A} \qquad (3)$$

We clean up the first equation:

$$v_1 \frac{1}{50\ \text{k}\Omega} + v_2 \frac{1}{30\ \text{k}\Omega} - v_3 \frac{1}{50\ \text{k}\Omega} = -140\ \mu\text{A}$$

or, after multiplying by the common denominator, 150 kΩ:

$$3v_1 + 5v_2 - 3v_3 = -21\ \text{V} \qquad (1')$$

And we clean up the third equation:

$$-v_1 \frac{1}{50\ \text{k}\Omega} + v_3 \frac{1}{50\ \text{k}\Omega} + \frac{1}{55\ \text{k}\Omega} = 40\ \mu\text{A} + \frac{31\ \text{V}}{55\ \text{k}\Omega}$$

or, after multiplying by the common denominator, 550 kΩ:

$$-11v_1 + 21v_3 = 22 + 310\ \text{V} = 332\ \text{V} \qquad (3')$$

The simplest equation is Eqn (2), which we may solve for v_2:

$$v_2 = v_1 - 50\ \text{k}\Omega \frac{31\ \text{V} - v_3}{110\ \text{k}\Omega} = v_1 + \frac{5}{11} v_3 - \frac{155}{11}\ \text{V}$$

Substituting this into Eqns (1') yields the following:

$$3v_1 + 5\left(v_1 + \frac{5}{11} v_3 - \frac{155}{11}\ \text{V}\right) - 3v_3 = -21\ \text{V}$$

or, after multiplying by 11:

$$88v_1 - 8v_3 = -21(11)\ \text{V} + 775\ \text{V} = 544\ \text{V}$$

or, after dividing by 8:

$$11v_1 - v_3 = 68\ \text{V} \qquad (1'')$$

Since v_2 is absent from Eqn (3'), we proceed to solve (1'') and (3') for v_3. (We solve for v_3 because v_1 is easier to eliminate.) We use (1'') + (3'):

$$\begin{array}{rl} -11v_1 + 21v_3 &= 332\ \text{V} \\ +\ \ 11v_1 - \ \ \ v_3 &= \ \ 68\ \text{V} \\ \hline = \qquad\qquad 20v_3 &= 400\ \text{V} \end{array}$$

or

$$v_3 = 20\text{ V}$$

Using (1'') we find the value of v_1 from v_3:

$$11v_1 = 68\text{ V} + v_3 = 68\text{ V} + 20\text{ V} = 88\text{ V}$$

or

$$v_1 = 8\text{ V}$$

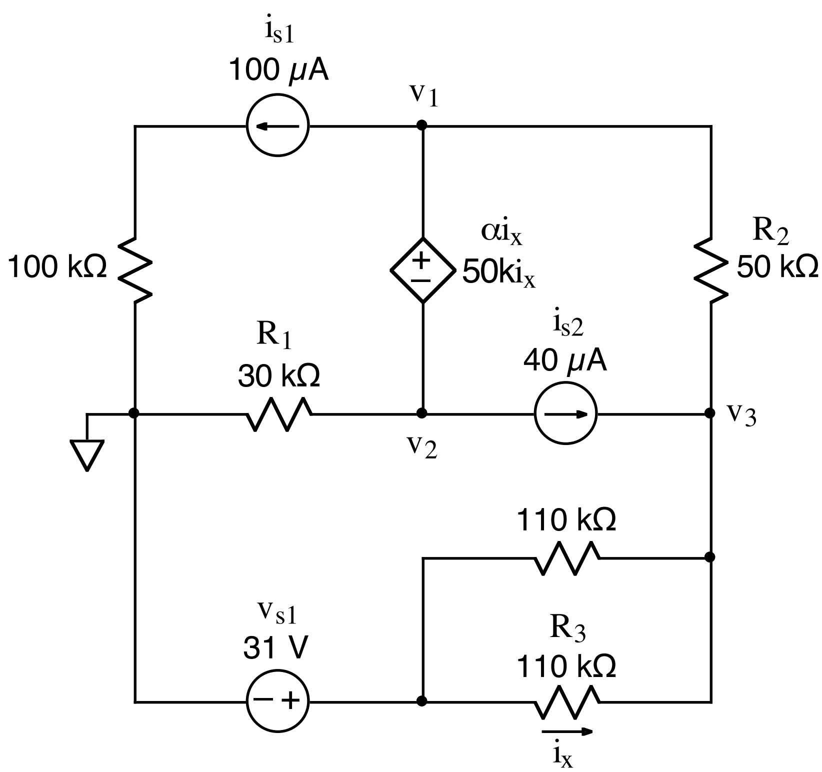
Finally, we use (1') to find v_2:

$$5v_2 = -21\text{ V} - 3v_1 + 3v_3 = -21\text{ V} - 3(8\text{ V}) + 3(20\text{ V}) = 15\text{ V}$$

or

$$v_2 = \frac{15\text{ V}}{5} = 3\text{ V}$$

An alternative approach is to find a symbolic solution. We start by labeling the components in the circuit, as shown below:



Using conductance, (i.e., 1/resistance), we have the following symbolic equations:

$$i_{s1} + (v_1 - v_3)g_2 + v_2 g_1 + i_{s2} = 0 \text{ A}$$

$$v_1 - v_2 = \alpha(v_{s1} - v_3)g_3$$

$$(v_3 - v_1)g_2 - i_{s2} + (v_3 - v_{s1})2g_3 = 0 \text{ A}$$

In matrix form, we have the following equation:

$$\begin{bmatrix} g_2 & g_1 & -g_2 \\ 1 & -1 & \alpha g_3 \\ -g_2 & 0 & g_2 + 2g_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -(i_{s1} + i_{s2}) \\ \alpha v_{s1} g_3 \\ i_{s2} + 2v_{s1} g_3 \end{bmatrix}$$

If we multiply the second row by g_2, we have a convenient form for eliminating v_1:

$$\begin{bmatrix} g_2 & g_1 & -g_2 \\ g_2 & -g_2 & \alpha g_2 g_3 \\ -g_2 & 0 & g_2 + 2g_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -(i_{s1} + i_{s2}) \\ \alpha g_2 g_3 v_{s1} \\ i_{s2} + 2v_{s1} g_3 \end{bmatrix}$$

We add the first and third rows, and we add the second and third rows to eliminate v_1. The result is a smaller matrix equation:

$$\begin{bmatrix} g_1 & 2g_3 \\ -g_2 & \alpha g_2 g_3 + g_2 + 2g_3 \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -i_{s1} + 2v_{s1} g_3 \\ \alpha g_2 g_3 v_{s1} + i_{s2} + 2v_{s1} g_3 \end{bmatrix}$$

Now can multiply the first row by g_2 and the second row by g_1 to obtain terms in the first column that will cancel out when the rows are summed:

$$\begin{bmatrix} g_1 g_2 & 2g_2 g_3 \\ -g_1 g_2 & [\alpha g_2 g_3 + g_2 + 2g_3]g_1 \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} [-i_{s1} + 2v_{s1} g_3]g_2 \\ [\alpha g_2 g_3 v_{s1} + i_{s2} + 2v_{s1} g_3]g_1 \end{bmatrix}$$

Summing the first two rows yields an equation for v_3:

$$\{[\alpha g_2 g_3 + g_2 + 2g_3]g_1 + 2g_2 g_3\}v_3 = [\alpha g_2 g_3 v_{s1} + i_{s2} + 2v_{s1} g_3]g_1 + [-i_{s1} + 2v_{s1} g_3]g_2$$

or

$$v_3 = \frac{[\alpha g_2 g_3 v_{s1} + i_{s2} + 2v_{s1} g_3]g_1 + [-i_{s1} + 2v_{s1} g_3]g_2}{\{[\alpha g_2 g_3 + g_2 + 2g_3]g_1 + 2g_2 g_3\}}$$

This result looks daunting, but we can rearrange things somewhat:

$$v_3 = \frac{v_{s1}[(\alpha g_2 + 2)g_1 g_3 + 2g_2 g_3] - i_{s1} g_2 + i_{s2} g_1}{(\alpha g_2 + 2)g_1 g_3 + g_1 g_2 + 2g_2 g_3}$$

Multiplying the top and bottom by $R_1 R_2 R_3$ simplifies the form somewhat:

$$v_3 = \frac{v_{s1}[(\alpha g_2 + 2)R_2 + 2R_1] - i_{s1} R_1 R_3 + i_{s2} R_2 R_3}{(\alpha g_2 + 2)R_2 + R_3 + 2R_1}$$

Now we calculate the various terms:

$$\alpha g_2 + 2 = 50\ \text{k}\Omega \cdot \frac{1}{50\ \text{k}\Omega} + 2 = 1 + 2 = 3$$

$$i_{s1} R_1 R_3 = 100\ \mu\text{A} \cdot 30\ \text{k}\Omega \cdot 110\ \text{k}\Omega = 3\ \text{V} \cdot 110\ \text{k}\Omega$$

$$i_{s2} R_2 R_3 = 40\ \mu\text{A} \cdot 50\ \text{k}\Omega \cdot 110\ \text{k}\Omega = 2\ \text{V} \cdot 110\ \text{k}\Omega$$

Substituting for all terms, we obtain a value for v_3:

$$v_3 = \frac{31\ \text{V}[3 \cdot 50\ \text{k}\Omega + 2 \cdot 30\ \text{k}\Omega] - 3\ \text{V} \cdot 110\ \text{k}\Omega + 2\ \text{V} \cdot 110\ \text{k}\Omega}{3 \cdot 50\ \text{k}\Omega + 110\ \text{k}\Omega + 2 \cdot 30\ \text{k}\Omega}$$

or

$$v_3 = \frac{31\ \text{V}[210\ \text{k}\Omega] - 1\ \text{V} \cdot 110\ \text{k}\Omega}{320\ \text{k}\Omega} = \frac{31\ \text{V}[21] - 1\ \text{V} \cdot 11}{32}$$

or

$$v_3 = \frac{640\ \text{V}}{32} = 20\ \text{V}$$

From the top row of the first 2 x 2 matrix equation, above, we obtain an equation relating v_2 and v_3:

$$g_1 v_2 + 2g_3 v_3 = -i_{s1} + 2v_{s1} g_3$$

$$\begin{bmatrix} g_1 & 2g_3 \\ -g_2 & \alpha g_2 g_3 + g_2 + 2g_3 \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -i_{s1} + 2v_{s1} g_3 \\ \alpha g_2 g_3 v_{s1} + i_{s2} + 2v_{s1} g_3 \end{bmatrix}$$

or

$$v_2 = \frac{-i_{s1} + 2v_{s1}g_3 - 2g_3v_3}{g_1}$$

Multiplying top and bottom by R_1R_3, we have the following result:

$$v_2 = \frac{-i_{s1}R_1R_3 + 2v_{s1}R_1 - 2R_1v_3}{R_3}$$

or

$$v_2 = \frac{-100\ \mu\text{A} \cdot 30\ \text{k}\Omega \cdot 110\ \text{k}\Omega + 2 \cdot 31\ \text{V} \cdot 30\ \text{k}\Omega - 2 \cdot 30\ \text{k}\Omega \cdot 20\ \text{V}}{110\ \text{k}\Omega}$$

or

$$v_2 = \frac{-3\ \text{V} \cdot 110\ \text{k}\Omega + 62\ \text{V} \cdot 30\ \text{k}\Omega - 40\ \text{V} \cdot 30\ \text{k}\Omega}{110\ \text{k}\Omega}$$

or

$$v_2 = \frac{-3\ \text{V} \cdot 11 + 62\ \text{V} \cdot 3 - 40\ \text{V} \cdot 3}{11} = \frac{3\ \text{V} \cdot 11}{11} = 3\ \text{V}$$

Finally, we return to our original equation for the supernode voltage relationship to find v_1:

$$v_1 - v_2 = \alpha(v_{s1} - v_3)g_3$$

or

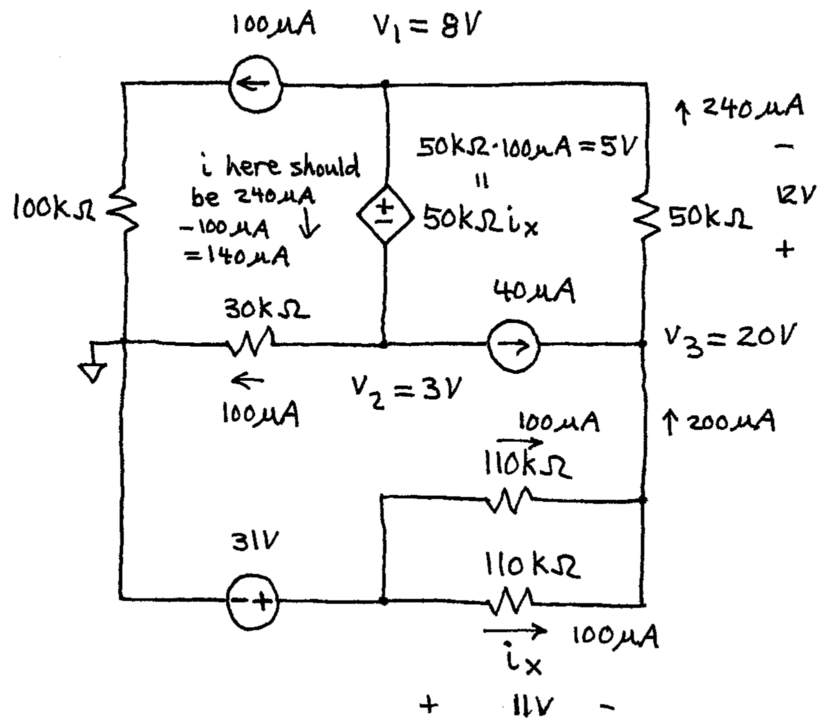
$$v_1 = 50\ \text{k}\Omega(31\ \text{V} - 20\ \text{V})/110\ \text{k}\Omega + 3\ \text{V} = 8\ \text{V}$$

To verify the above answers, we perform a consistency check. Using the node voltages, we calculate the currents flowing in the circuit and check to see if they sum to zero at each node.

We find i_x using v_3 and the 31 V source.

$$i_x = \frac{31\ \text{V} - 20\ \text{V}}{110\ \text{k}\Omega} = \frac{11\ \text{V}}{110\ \text{k}\Omega} = 100\ \mu\text{A}$$

We find the other currents by using Ohm's law. The diagram below shows that the currents at every node sum to zero. It follows that our solution is correct.



100μA
V1 = 8V
↑ 240μA
50kΩ·100μA = 5V
=
50kΩ ix
i here should
be 240μA
−100μA
=140μA
100kΩ
50kΩ
−
12V
+
30kΩ
40μA
V3 = 20V
100μA
V2 = 3V
100μA
110kΩ
↑ 200μA
31V
110kΩ
100μA
ix
+
11V
−