Tuesday 19th HW 7

Tuesday 26th by 5pm HW 8

Bring Lab 1 Intro/Abstract to me on Thursday 21st class.
(lab 1 introduction)

Unit 4

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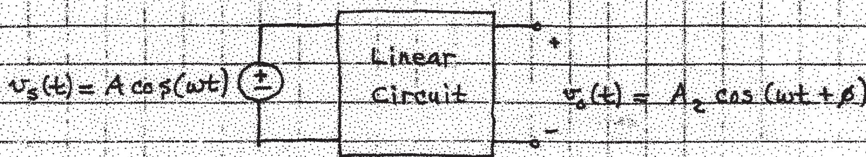
19 Mar 01
Neil E Cotter

Sinusoidal Signals



- input sinusoid, output also sinusoid!
  freq unchanged!

  amplitude & phase shift change
  calc these

- Can write arbitrary input (periodic) as sum of sinusoids of different frequencies

Imaginary #'s What is j?

9 0) j allows to find roots of all polynomials

8 1) $j = \sqrt{-1}$

7 2) j is a unit vector ⊥ to real axis

6 3) j defines rule for multiplying vecs

5 4) $j = 90°$ rotation

cart 3 4 5) j allows us to use phasors for $A\cos(\omega t + \phi)$
re

2 6) j is a token placed in front of sin() to make cos()

1 7) j allows us to write $\cos(\omega t + \phi)$ as $\boxed{a + jb}$ = cos + sin
w/no phase shift

combine 8) j allows us to use complex exp for polar form $Ae^{j\theta}$ draw rect polar pict

10 9) j results from taking $\frac{d}{dt} e^{j\phi(t)} = j e^{j\phi(t)} \frac{d\phi(t)}{dt}$ L's and C's

✓4 10) j allows us to write $\cos = \frac{e^{j} + e^{-j}}{2}$

Complex #'s:

$a + bj$ ← used so that current (i symbol) not get confused

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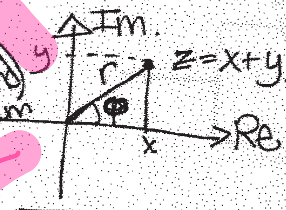
a = real part, b = imaginary part

complex plane or (Argand) diagram



$z = x + yj$ ← Rectangular or Cartesian form

polar form ~~or~~ exponential form

$z = r e^{i\varphi}$ where $\varphi = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & x>0 \\ \tan^{-1}\left(\frac{y}{x}\right)+\pi & x<0,\ y\geq 0 \\ \tan^{-1}\left(\frac{y}{x}\right)-\pi & x<0,\ y<0 \\ +\pi/2 & x=0,\ y>0 \\ -\pi/2 & x=0,\ y<0 \end{cases}$

$r = \sqrt{x^2+y^2}$

trigonometric form:

(phi)

$z = r(\cos\varphi + j\sin\varphi) = \underbrace{r\cos\varphi}_{a} + \underbrace{r\sin\varphi}_{b} j$

Suggestions ⇒

add/sub. using rect. form → $a+bj + (c+dj) = (a+c) + (b+d)j$

mult./div. " exponential form

$(r_1 e^{i\varphi_1}) * (r_2 e^{i\varphi_2}) = r_1 \cdot r_2 e^{i(\varphi_1+\varphi_2)}$

$\frac{e^{i\varphi_1}}{e^{i\varphi_2}} = e^{i(\varphi_1-\varphi_2)}$

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EX: Express $3e^{j32°} + 4e^{-j40°}$ in rectangular and polar form.

ANS: $5.608 - j0.981$ or $5.693e^{-j9.922°}$ (approx)

SOL'N: Use Euler's formula to write each complex number in rectangular form $a + jb$:

$$Ae^{j\phi} = A\cos\phi + jA\sin\phi$$

$$3e^{j32°} = 3\cos(32°) + j3\sin(32°) = 2.544 + j1.590$$

$$4e^{-j40°} = 4\cos(-40°) + j4\sin(-40°) = 3.064 - j2.571$$

Sum the real and imaginary parts:

$$3e^{j32°} + 4e^{-j40°} = 2.544 + 3.064 + j(1.590 - 2.571)$$

Our answer in rectangular form:

$$3e^{j32°} + 4e^{-j40°} = \boxed{5.608 - j0.981}$$

Use the Pythagorean theorem to find the magnitude for polar form:

$$\left|3e^{j32°} + 4e^{-j40°}\right| = \sqrt{5.608^2 + 0.981^2} = 5.693$$

Use tangent of phase angle = Im/Re to find angle for polar form:

$$\angle\left(3e^{j32°} + 4e^{-j40°}\right) = \tan^{-1}\left(\frac{-0.981}{5.608}\right) = -9.922°$$

Our answer in polar form:

$$3e^{j32°} + 4e^{-j40°} = 5.693e^{-9.922°}$$

EX: Rationalize $\dfrac{3+j1.5}{(5-j2.6)}$

ANS: $0.35+j0.48$ $(5-j2.6)^* = (5+j2.6)$

SOL'N: Multiply the numerator and denominator by the denominator's complex conjugate. This turns the denominator into a real number equal to the magnitude squared of the original denominator.

$$\frac{3+j1.5}{5-j2.6} = \frac{3+j1.5}{5-j2.6}\frac{(5+j2.6)}{(5+j2.6)} \quad j^2=-1$$

$$= \frac{(3+j1.5)(5+j2.6)}{5^2+2.6^2}$$

$$= \frac{15-3.9+j(7.8+7.5)}{31.76}$$

$$= 0.35+j0.48$$

OR

$$\frac{3+j(1.5)}{5-j2.6} = \frac{\sqrt{9+1.5^2}\,e^{j\tan^{-1}\left(\frac{1.5}{3}\right)}}{\sqrt{25+2.6^2}\,e^{-j\tan^{-1}\left(\frac{2.6}{5}\right)}}$$

$$= \frac{3.354}{5.636}e^{j(26.6^\circ+27.5)} = 0.6e^{j(54.1^\circ)}$$

$$= 0.6\cos(54.1^\circ) + 0.6\sin(54.1^\circ)j$$

$$= 0.35 + 0.49j$$

EX: Evaluate $\dfrac{4+j3}{2-j6}$

ANS: $-\dfrac{1}{4}+j\dfrac{3}{4}$

SOL'N: Multiply the numerator and denominator by the denominator's complex conjugate. This turns the denominator into a real number equal to the magnitude squared of the original denominator. Divide the numerator's real and imaginary parts by this new denominator.

$$\frac{4+j3}{2-j6} = \frac{4+j3}{2-j6}\,\frac{(2-j6)^*}{(2-j6)^*}$$

$$= \frac{4+j3}{2-j6}\,\frac{2+j6}{2+j6}$$

$$= \frac{8-18+j(6+24)}{4+36}$$

$$= \frac{-10+j30}{40}$$

$$= -\frac{1}{4}+j\frac{3}{4}$$

OR

$$\frac{\sqrt{16+9}\,e^{j\tan^{-1}(3/4)}}{\sqrt{4+36}\,e^{-j\tan^{-1}(6/2)}} = \frac{5}{6.32}e^{j(37^\circ+72^\circ)} = 0.8e^{j(109^\circ)}$$

$$= +0.8\cos(109^\circ)+0.8\sin(109^\circ)j = -.26+0.76j$$

DEF: Complex Conjugate of $a + jb \equiv a - jb \equiv$ complex number with imaginary part inverted

NOT'N: $z^* \equiv$ complex conjugate of z

TOOL: To find the complex conjugate of an expression, change each j to $-j$.

NOTE: This is equivalent to (but easier than) converting the expression to form $a + jb$ and changing it to $a - jb$.

TOOL: $\left(Ae^{j\phi}\right)^* = Ae^{-j\phi}$ when A is real

TOOL: To find z^*, reflect z around the real axis. In other words, preserve the magnitude but take the negative of the phase angle.

EX: Evaluate $(2 + j3)^*$

ANS: $2 - j3$

SOL'N: This is a direct application of the definition of complex conjugate.

EX: Evaluate $\left(\mathrm{Re}^{j\psi}\right)^*$ where R is real

ANS: $\mathrm{Re}^{-j\psi}$

SOL'N: We retain the magnitude but invert the phase angle to find the conjugate of a complex number in polar form.

EX: Evaluate $\left[\dfrac{(6 - j2)e^{j35°}}{(4 + j5)\sin(x)}\right]^*$ where x is real

ANS: $\left[\dfrac{(6 + j2)e^{-j35°}}{(4 - j5)\sin(x)}\right]$

SOL'N: Change each j to $-j$. Thus, $-j$ becomes $-(-j) = j$.

EX: Find $\mathrm{Re}[7e^{j25°}]$, (i.e., find the real part)

ANS: $7\cos(25°) \approx 6.34$

SOL'N: The answer follows directly from Euler's formula (see tools for rectangular and polar forms for complex numbers):

$$Ae^{j\phi} = A\cos\phi + jA\sin\phi$$

EX: Find $\left|2e^{j182°}\right|$, (i.e., find the magnitude)

ANS: 2

SOL'N: The magnitude of a product is the product of the magnitudes:

$$\left|2e^{j182°}\right| = |2| \cdot \left|e^{j182°}\right|$$

The magnitude of a real number is the absolute value of that real number:

$$|2| = 2$$

The magnitude of e^{jx} for any real x is 1:

$$\left|e^{jx}\right| = 1$$

Thus we have:

$$\left|e^{j182°}\right| = 1$$

Putting our results together gives the answer:

$$\left|2e^{j182°}\right| = 2$$

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EX: Find $\text{Re}[\frac{6+j3}{2-j4}e^{jx}]$, (i.e., find the real part) where "x" is real

ANS: $1.5\cos(x+\pi/2)$

SOL'N: We may take one of several different approaches to convert the quantity inside the brackets into the form $a + jb$ (where a is our final answer). We'll take the approach of rationalizing the fraction.

$$\text{Re}[\frac{6+j3}{2-j4}e^{jx}] = \text{Re}[\frac{6+j3}{2-j4}\frac{2+j4}{2+j4}e^{jx}]$$

$$= \text{Re}[\frac{12-12+j(24+6)}{2^2+4^2}e^{jx}]$$

$$= \text{Re}[\frac{j30}{20}e^{jx}]$$

We now use Euler's formula to expand the complex exponential:

$$= \text{Re}[\frac{j30}{20}\{\cos(x)+j\sin(x)\}]$$

$$= \text{Re}[-1.5\sin(x)+j1.5\cos(x)]$$

Our final answer is the real part, which we may express in several ways.

$$\text{Re}[\frac{6+j3}{2-j4}e^{jx}] = -1.5\sin(x) \text{ or}$$

$$\text{Re}[\frac{6+j3}{2-j4}e^{jx}] = 1.5\cos(x+\pi/2) = 1.5\cos(x+90°)$$

NOTE: A curious feature of this problem is that the fraction consisting of complex numbers is purely imaginary. We now examine this symbolically.

$$k\cdot\frac{a+jb}{b-ja} = k\cdot\frac{j(b-ja)}{b-ja} = jk$$

Whenever the numerator and denominator of a fraction have the above pattern, we will find that the result is purely imaginary. Note the necessary minus sign.

EX: Find the value (in polar form) of $(6+j5)^{1/5}$.

ANS: $1.51e^{j7.96°}$ (approx)

SOL'N: First, we convert the number being raised to a power to polar form:

$$6+j5=\sqrt{6^2+5^2}e^{j\tan^{-1}\left(\frac{5}{6}\right)}\approx 7.81e^{j39.8°}$$

Now take the power inside the parentheses and use the identity

$$(ab)^n=a^nb^n$$

giving the answer:

$$(6+j5)^{1/5}\approx 7.81^{1/5}e^{j39.8°/5}\approx 1.51e^{j7.96°}$$

EX: Find the value (in polar form) of $\left(3e^{j40°}\right)^3$.

ANS: $27e^{j120°}$

SOL'N: We take the power inside the parentheses and use the identity

$$(ab)^n=a^nb^n$$

giving the answer:

$$3^3e^{j3\cdot 40°}=27e^{j120°}$$

EX: Find the polar form of $2.5 - j3.2$.

ANS: $4.06e^{-j52°}$

SOL'N: We express $2.5 - j3.2$ in polar form $Ae^{j\phi}$.

Use the pythagorean theorem to find magnitude A:

$$A = \sqrt{2.5^2 + 3.2^2} \approx 4.06$$

Set the tangent of the phase angle equal to the side opposite (imaginary part) over the side adjacent (real part):

$$\tan\phi = \frac{\text{Im}[2.5 - j3.2]}{\text{Re}[2.5 - j3.2]} = \frac{-3.2}{2.5} = -1.28$$

$$\phi = \tan^{-1}\left(\frac{-3.2}{2.5}\right) \approx -52° \text{ or } -0.9076 \text{ radians}$$

Our final answer:

$$2.5 - j3.2 \approx 4.06e^{-j52°}$$

NOTE: When calculating the inverse tangent, if we use -1.28 rather than both the imaginary and real parts, we have two possible values for ϕ that differ by 180 degrees. The ratio of the imaginary and real parts is the same for $1 + j$ and $-1 - j$, for example. Thus, it is necessary to keep track of which quadrant the complex number lies in if we wish to avoid confusion about the correct value of phase angle ϕ.

EX: Find the rectangular form of $6e^{-j47°}$

ANS: $4.09 - j4.39$

SOL'N: We must express $6e^{-j47°}$ in rectangular form $a + jb$.

We use Euler's formula for the complex exponential:

$$6e^{-j47°} = 6\cos(-47°) + j6\sin(-47°)$$

Applying identities, $\cos(-A) = \cos(A)$ and $\sin(-A) = -\sin(A)$, we have

$$= 6\cos(47°) - j6\sin(47°)$$

$$6e^{-j47°} = 4.09 - 4.39$$

Ex: Find the rectangular and polar forms of $\dfrac{6-j2}{4+j5}$.

Ans: $\dfrac{14}{41}-j\dfrac{38}{41}$ and $0.987e^{-j69.78°}$

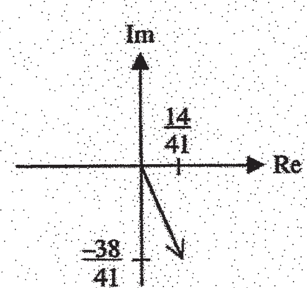
Sol'n:

$$\frac{6-j2}{4+j5}=\frac{6-j2}{4+j5}\,\frac{(4+j5)^*}{(4+j5)^*}=\frac{6-j2}{4+j5}\,\frac{(4-j5)}{(4-j5)}$$

$$=\frac{24-10-j(30+8)}{16+25}=\frac{14}{41}-j\frac{38}{41}$$

$$A=\sqrt{\left(\frac{14}{41}\right)^2+\left(\frac{38}{41}\right)^2}=0.987$$

$$\phi=\tan^{-1}\left(\frac{-38}{41}\right)=-69.78°$$



TUTOR: THE PHASOR TRANSFORM

All voltages and currents in linear circuits with sinusoidal sources are described by constant-coefficient linear differential equations of the form

$$(1) \qquad a_n \frac{d^n f}{dt^n} + a_{n-1} \frac{d^{n-1} f}{dt^{n-1}} + \ldots + a_0 f = C \cos(\omega t + \phi)$$

where f is a function of time, the a_n are constants, C is a constant, ω is the radian frequency of the sinusoidal source, and ϕ is the phase of the sinusoidal source. In (1), f represents any voltage or current in the circuit.

A particular solution to (1) can be found by an elegant procedure called the **phasor transform** method. This supplementary material outlines the mathematical basis of the method. The phasor transform is defined by

$$(2) \qquad f(t) = \mathrm{Re}\left[\mathbf{F}(\omega)e^{j\omega t}\right]$$

where $\mathbf{F}(\omega)$ is a function of ω called the **phasor transform** of f(t), and Re means the real part of the quantity in the brackets. $\mathbf{F}(\omega)$ is complex; it has a real and an imaginary part.

Two key mathematical relationships are used in finding a particular solution to (1). The first is

$$(3) \qquad \mathrm{Re}[W] = \frac{W + W^*}{2}$$

where W is any complex number and W* is the complex conjugate of W. Using (3) with (2) gives

$$(4) \qquad f = \frac{\mathbf{F}e^{j\omega t} + \mathbf{F}^* e^{-j\omega t}}{2}$$

where f has been written for f(t) and **F** for $\mathbf{F}(\omega)$ for brevity. Note that **F** is **not** a function of time. The second relationship is

$$\text{(5)} \qquad \cos(\omega t+\phi)=\frac{e^{j(\omega t+\phi)}+e^{-j(\omega t+\phi)}}{2}$$

which is called Euler's formula.

Substituting (5) and (4) into (1), taking the derivatives with respect to time, and collecting terms gives

$$\text{(6)} \qquad \left[a_n(j\omega)^n\mathbf{F}+a_{n-1}(j\omega)^{n-1}\mathbf{F}+\ldots+a_0\mathbf{F}-Ce^{j\phi}\right]e^{j\omega t}+$$
$$\left[a_n(-j\omega)^n\mathbf{F}^*+a_{n-1}(-j\omega)^{n-1}\mathbf{F}^*+\ldots+a_0\mathbf{F}^*-Ce^{-j\phi}\right]e^{-j\omega t}=0.$$

Now because $e^{j\omega t}$ and $e^{-j\omega t}$ are **linearly independent** functions (see, for example, C. R. Wylie, *Advanced Engineering Mathematics*, 3rd ed., New York: McGraw-Hill, 1966, p. 444), (6) can be true for all time only if

$$\text{(7)} \qquad \left[a_n(j\omega)^n\mathbf{F}+a_{n-1}(j\omega)^{n-1}\mathbf{F}+\ldots+a_0\mathbf{F}-Ce^{j\phi}\right]=0$$

and

$$\text{(8)} \qquad \left[a_n(-j\omega)^n\mathbf{F}^*+a_{n-1}(-j\omega)^{n-1}\mathbf{F}^*+\ldots+a_0\mathbf{F}^*-Ce^{-j\phi}\right]=0.$$

Equations (7) and (8) are identical because one is the complex conjugate of the other, so only one is needed. An expression for **F** from (7) is

$$\text{(9)} \qquad \mathbf{F}=\frac{Ce^{j\phi}}{a_n(j\omega)^n+a_{n-1}(j\omega)^{n-1}+\ldots+a_0}.$$

A particular solution to (1) can now be obtained from (9) and (2):

$$\text{(10)} \qquad f=\mathrm{Re}\left[\mathbf{F}e^{j\omega t}\right]=\mathrm{Re}\left[\frac{Ce^{j\phi}e^{j\omega t}}{a_n(j\omega)^n+a_{n-1}(j\omega)^{n-1}+\ldots+a_0}\right].$$

Symbolically, the notation for a phasor transformation is

(11) $$\mathbf{P}[f(t)] = \mathbf{F}(\omega)$$

where the bold **P** means "phasor transform of". Thus, **F** is the phasor transform of f. Taking the derivative of both sides of (2) gives

$$\frac{df}{dt} = \mathrm{Re}\left[j\omega\mathbf{F}(\omega)e^{j\omega t}\right]$$

which corresponds to

$$\mathbf{P}\left[\frac{df}{dt}\right] = j\omega\mathbf{F}.$$

Similarly,

(12) $$\mathbf{P}\frac{d^n f}{dt^n} = (j\omega)^n\mathbf{F}$$

and

(13) $$\mathbf{P}\left[\cos(\omega t + \phi)\right] = e^{j\phi}$$

because

(14) $$\cos(\omega t + \phi) = \mathrm{Re}\left[e^{j\phi}e^{j\omega t}\right].$$

From the basic relation in (2) it can also be shown that

(15) $$\mathbf{P}\left[f_1 + f_2\right] = F_1 + F_2$$

and

(16) $\mathbf{P}[af] = a\mathbf{F}$

where

$$\mathbf{P}[f_1] = \mathbf{F}_1$$

and

$$\mathbf{P}[f_2] = \mathbf{F}_2$$

and "a" is a constant. The relation in (15) means that the phasor transform of a sum of functions can be found by taking the transform of each function and adding the transforms.

Equations (11), (12), (13), (15), and (16) describe phasor transforms. An inverse phasor transform relation is written as

(17) $f(t) = \mathbf{P}^{-1}[\mathbf{F}(\omega)]$.

Equations (11) and (17) are called a **transform pair**. Equation (11) states how to get **F** when f is known; (17) how to get f when **F** is known. Equation (2) is the inverse transform relation. The transform relation is derived as follows. f(t) will always be a sinusoid, because it is a particular solution to (1). Thus f(t) can be written as

(18) $f(t) = f_m \cos(\omega t + \alpha)$.

Substituting (18) into (2), using Euler's formula and (3) gives

$$\frac{f_m e^{j\omega t} e^{j\alpha} + f_m e^{-j\omega t} e^{-j\alpha}}{2} = \frac{\mathbf{F} e^{j\omega t} + \mathbf{F}^* e^{-j\omega t}}{2}.$$

Collecting terms and using the linear independence of $e^{j\omega t}$ and $e^{-j\omega t}$, as before, gives

(19) $\mathbf{F} = f_m e^{j\alpha}$

so the phasor transform of $f_m \cos(\omega t + \alpha)$ is $f_m e^{\alpha t}$. The transform pairs are thus

(20) $\mathbf{F} = \mathbf{P}[f] = f_m e^{j\alpha}$

and

(21) $f = \mathbf{P}^{-1}[\mathbf{F}] = \mathrm{Re}\left[\mathbf{F}e^{j\omega t}\right].$

With the phasor transform relations given in (12), (15, (16), (20), and (21), a particular solution to (1) can be found without going through the detailed derivation using (3) and linear independence. The phasor transform of (1) is taken term-by-term using (12), (13), (15), and (16) to get (7), which is then solved for **F**. Having found **F**, f is found by taking the inverse transform according to (21).

EX: Let's find a particular solution to

(22) $$\frac{d^3 f}{dt^3} + 3\frac{d^2 f}{dt^2} + 50\frac{df}{dt} - 60f = 500\cos(10t + \pi/3).$$

Taking the phasor transform of this equation gives

$$(j10)^3\mathbf{F} + 3(j10)^2\mathbf{F} + 50(j10)\mathbf{F} - 60\mathbf{F} = 500e^{j\pi/3}.$$

Solving for **F**,

$$\mathbf{F} = \frac{500e^{j\pi/3}}{-j100 - 300 + j500 - 60}.$$

Converting **F** to polar form gives

$$\mathbf{F} = 0.812e^{-j174.25^\circ}$$

and finding the inverse transform gives

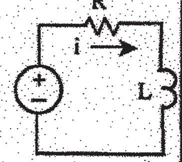
$$f = \mathrm{Re}\left[\mathbf{F}e^{j\omega t}\right] = \mathrm{Re}\left[0.812e^{-j174.25^\circ}e^{j10t}\right] = 0.812\cos(10t - 174.25^\circ).$$

COMMENT: The phasor transform method is powerful because it transforms a differential equation (1) into an algebraic equation (7), which can be solved for the phasor **F**, and then f can be found by taking the inverse transform.

Phasor voltages and currents satisfy Kirchhoff's laws, because of (15). Consequently, circuits can be transformed into the frequency domain, eliminating the need to write differential equations in the time domain and solve them by phasor transforms. The procedure for analyzing and designing circuits by transforming them into the frequency domain is summarized in the figure below. Note that impedance is defined as the ratio of a phasor voltage to a phasor current. Impedance is <u>not</u> defined in the time domain.

TIME DOMAIN

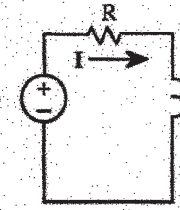
$v_g = v_m \cos(\omega t + \phi)$



phasor transform

FREQUENCY DOMAIN

$\mathbf{V}_g = v_m e^{j\phi}$



$R \to R$

$L \to j\omega L$

$C \to \dfrac{1}{j\omega C}$

Frequency-domain representation of the circuit (transformed circuit)

Apply circuit laws to obtain algebraic equations:

$$\mathbf{V}_g = R\,\mathbf{I} + j\omega L\,\mathbf{I}$$

Solve for the desired quantities:

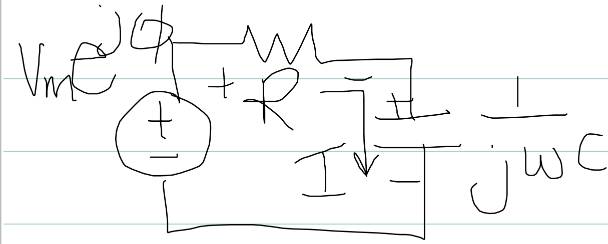
$$\mathbf{I} = \frac{\mathbf{V}_g}{R + j\omega L} = \frac{V_m e^{j\phi}}{\left(\sqrt{R^2+\omega^2 L^2}\right) e^{j\tan^{-1}\left(\frac{\omega L}{R}\right)}}$$

Inverse phasor transform

$$i = \frac{v_m \cos(\omega t + \phi - \psi)}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\psi = \tan^{-1}(\omega L/R)$$

Time-domain expression for the desired quantities



$\omega = 10$

$$+V_m e^{j\phi} - I\left(R + \frac{1}{j\omega C}\right) = 0$$

$$I = \frac{V_m e^{j\phi}}{R + \frac{1}{j\omega C}} = \frac{V_m e^{j\phi}}{\sqrt{R^2 + \left(\frac{-1}{\omega C}\right)^2}\, e^{j\tan^{-1}\left(\frac{-1}{\omega CR}\right)}}$$

Note: $\frac{1}{j\omega C} = \frac{-j}{\omega C}$

$$i(t) = \frac{V_m \cos\left(10t - \phi - \tan^{-1}\left(\frac{-1}{\omega CR}\right)\right]}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}$$

EX: If $f(t) = 2\sin(\omega t + \pi/3)$ find $\mathbf{P}[f(t)]$, (i.e., find the phasor)

ANS: $\mathbf{P}[f(t)] \equiv \mathbf{F} = 2e^{-j\pi/6} \equiv 2\angle\pi/6$

SOL'N: If we have a cosine, we use the standard identity for phasors:

$$\mathbf{P}[A\cos(\omega t + \phi)] = Ae^{j\phi} \equiv A\angle\phi$$

For a sine, we multiply the standard identity by $-j$ (which is the phasor for a sine of magnitude one and zero phase shift):

$$\mathbf{P}[\sin(\omega t)] = -j \equiv 1\angle -90°$$

Thus, we have

$$\mathbf{P}[f(t)] \equiv \mathbf{F} = -2je^{j\pi/3}.$$

The above is mathematically correct and works properly in solving problems, but we will apply identities to express the answer in standard form:

$$-1 = e^{j180°} = e^{-j180°} = e^{j\pi} = e^{-j\pi}.$$

NOTE: (We use whichever of +180° or –180° is most convenient.)

$$j = e^{j90°} = e^{j\pi/2}.$$

Applying the identities:

$$\mathbf{F} = -2je^{j\pi/3} = 2e^{-j\pi}e^{j\pi/2}e^{j\pi/3} = 2e^{-j\pi/6} \equiv 2\angle -\pi/6.$$

EX: If $\mathbf{F} = (2.5 + j3.2)$ find $\mathbf{P}^{-1}[\mathbf{F}]$, (i.e., find the inverse phasor)

ANS: $\mathbf{P}^{-1}[\mathbf{F}] = 4.06\cos(\omega t + 52°)$

SOL'N: We convert to polar form:

$$2.5 + j3.2 = \sqrt{2.5^2 + 3.2^2}\,e^{j\tan^{-1}\left(\frac{3.2}{2.5}\right)} \approx 4.06e^{j52°}$$

Now use the standard inverse phasor identity:

$$\mathbf{P}^{-1}[Ae^{j\phi}] = A\cos(\omega t + \phi)$$

NOTE: There is no math to do here—we just substitute the values of A and ϕ into the cos( ).

NOTE: We don't know the value of ω for this problem. Thus, we just use a symbolic variable for ω. The value of ω is *not* part of a phasor. (The value of ω must be kept track of separately.)

Using the identity gives the answer:

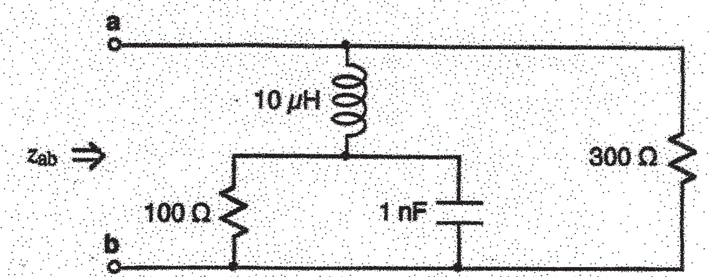
$$\mathbf{P}^{-1}[\mathbf{F}] = 4.06\cos(\omega t + 52°)$$

NOTE: Mathematically, it is also correct to invert the given phasor in two pieces, with the real part giving a cosine term having no phase shift and the imaginary part giving a (negative) sine term having no phase shift:

$$\mathbf{P}^{-1}[2.5 + j3.2] = 2.5\cos(\omega t) - 3.2\sin(\omega t).$$

Although this answer is correct, it is usually easier to visualize a single sinusoid with a phase shift. The sum of the cos and sin terms is equal to the single cos with a phase shift given above. This follows from the observation that the sum of any number of sinusoids of the same frequency may be expressed as a single sinusoid of that frequency. (The challenging part is determining the magnitude and phase shift of the single sinusoid.)

Ex:

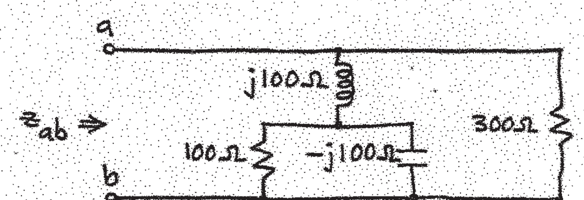


Given ω = 10 M rad/s, find z_{ab}.

Sol'n: $z_L = j\omega L = j\,10\cancel{M}\,r/s \cdot 10\cancel{\mu}H = j\,100\,\Omega$

$$z_C = \frac{-j}{\omega C} = \frac{-j}{10M\ r/s \cdot 1nF} = \frac{-j}{10m}\,\Omega = -j100\,\Omega$$

frequency domain (or s-domain) model:



We first consider the *R* and *C* in parallel.

$$100\,\Omega \parallel -j100\,\Omega = 100\,\Omega \cdot 1 \parallel -j = 100\,\Omega \cdot \frac{-j}{1-j}$$

Rationalizing this expression, we have

$$100\,\Omega \parallel -j100\,\Omega = 100\,\Omega\ \frac{-j}{1-j} \cdot \frac{1+j}{1+j}$$

or

$$100\,\Omega \parallel -j100\,\Omega = 100\,\Omega \quad \frac{1-j}{2} = 100\,\Omega \cdot \left(\frac{1}{2} - \frac{j}{2}\right)$$

Now we add $z_L = j100\,\Omega$

$$100\,\Omega \parallel -j100\,\Omega + j100\,\Omega = 100\,\Omega \left(\frac{1}{2} - \frac{j}{2} + j\right)$$

$$= 100\,\Omega \left(\frac{1}{2} + \frac{j}{2}\right)$$

To find z_{ab}, we use conductance $g_{ab} = \frac{1}{z_{ab}}$:

$$g_{ab} = \frac{1}{100\,\Omega \parallel -j100\,\Omega + j100\,\Omega} + \frac{1}{300\,\Omega}$$

$$= \frac{1}{100\,\Omega \left(\frac{1}{2} + \frac{j}{2}\right)} + \frac{1}{100\,\Omega} \cdot \frac{1}{3}$$

$$= \frac{1}{100\,\Omega} \cdot \frac{2}{1+j} \cdot \frac{1-j}{1-j} + \frac{1}{100\,\Omega} \cdot \frac{1}{3}$$

$$= \frac{1}{100\,\Omega} \cdot \frac{2-j2}{2} + \frac{1}{100\,\Omega} \cdot \frac{1}{3}$$

$$= \frac{1}{100\,\Omega} (1-j) + \frac{1}{100\,\Omega} \cdot \frac{1}{3}$$

$$= \frac{1}{100\,\Omega} \left(\frac{4}{3} - j\right)$$

Now we calculate $\frac{1}{g_{ab}} = z_{ab}$.

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$$z_{ab} = \frac{1}{g_{ab}} = 100\,\Omega \frac{1}{\frac{4}{3} - j}$$

$$= 100\,\Omega \frac{3}{4 - j3}$$

$$= 100\,\Omega \frac{3}{4 - j3} \cdot \frac{4 + j3}{4 + j3}$$

$$= 100\,\Omega \frac{12 + j9}{4^2 + 3^2}$$

$$= 100\,\Omega \frac{12 + j9}{25}$$

$$= 4 \cdot (12 + j9)$$

$$\boxed{z_{ab} = 48 + j36}$$

Homework #7 Examples

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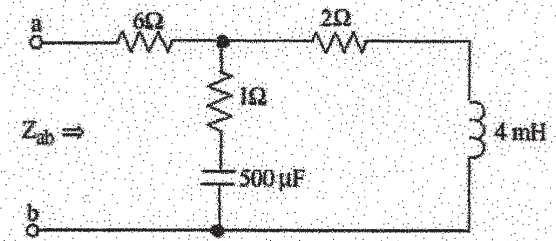
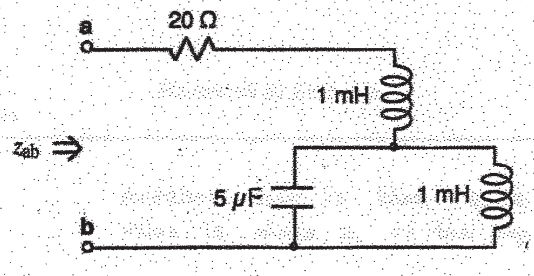
1.



Given $\omega = 1k$ rad/sec, find Z_{ab}.

Ex:



Find a frequency, ω, that causes z_{ab} to be real, (i.e., imaginary part equals zero).

Sol'n: $z_{ab} = 20\,\Omega + z_{L1} + z_C \| z_{L2}$

For z_{ab} to be real, we must have

$$z_{L1} + z_C \| z_{L2} = \text{real}$$

One simple sol'n is to let $\omega = 0$ so both L's act like wires and C acts like open circuit.

Other potential sol'ns are $\omega = \infty$, (so L's act like opens, resulting in $z_{ab} = \infty$), and ω = frequency where $z_C = -z_{L2}$, (so C and L in parallel have equal but opposite impedances).

The latter case, where $z_C = -z_L$ gives the interesting result that $z_C \| z_L = \frac{L/C}{0} = \infty$

This means $z_{ab} = \infty\,\Omega$. In this case, (unlike $\omega \to \infty$), $z_{ab} \to \infty$ along real axis as $z_C \| z_L \to \infty$.

or $\omega = 20\text{K r/s}$

Another sol'n is that $z_C \| z_L$ has a value is minus z_L of the top inductor.

In that case, $z_L + z_C \| z_L = 0$ and $z_{ab} = 0 =$ wire.

$$z_L = j\omega L$$

$$z_C \| z_L = \frac{-j}{\omega C} \| j\omega L = \frac{\frac{-j}{\omega C} \cdot j\omega L}{\frac{-j}{\omega C} + j\omega L} = \frac{L/C}{j(\omega L - \frac{1}{\omega C})}$$

$$= -j \frac{L/C}{\omega L - \frac{1}{\omega C}}$$

Thus, we want $j\omega L - \dfrac{jL/C}{\omega L - \frac{1}{\omega C}} = 0$

or $\omega L = \dfrac{L/C}{\omega L - \frac{1}{\omega C}}$

or $\omega L \left(\omega L - \frac{1}{\omega C} \right) = L/C$

or $\omega^2 L^2 - \dfrac{L}{C} = \dfrac{L}{C}$

or $\omega^2 L^2 = \dfrac{2L}{C}$ or $\omega^2 = \dfrac{2}{LC}$

or $\omega = \sqrt{\dfrac{2}{LC}}$ or $\omega = \sqrt{\dfrac{2}{5\mu F \cdot 1mH}}$

or $\omega = \sqrt{\dfrac{2}{5} G}$ r/s $= \sqrt{400M}$ r/s

or $\omega = 20k$ r/s

Homework #7 Examples

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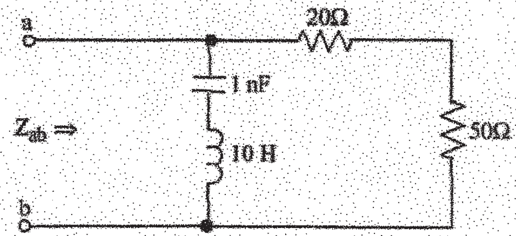
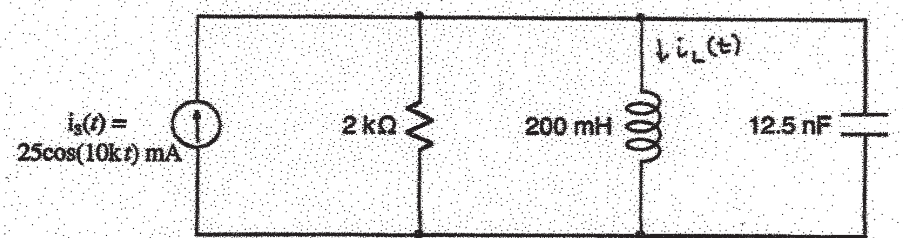
2.



Find a frequency, ω, that causes Z_{ab} to be real (i.e. imaginary part equals zero). $\omega \neq 0$ or $\omega \neq \infty$.

Ex:



a) Find the phasor value for $i_s(t)$.

b) Draw the frequency-domain circuit diagram, including the phasor value for $i_s(t)$ and impedance values for components.

c) Find the phasor value for $i_L(t)$.

Sol'n: a) The phasor for $A\cos(\omega t + \phi)$ is $Ae^{j\phi}$.

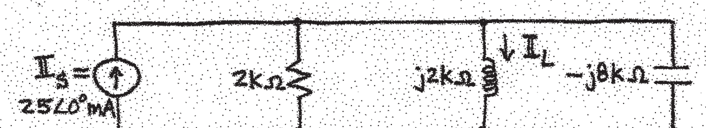
$$\therefore \mathbb{I}_s = 25\, e^{j0^\circ} \text{ mA or } 25 \angle 0^\circ \text{ mA}$$

b) From $i_s(t)$, we see that $\omega = 10\text{k rad/s}$.

Impedance $z_L = j\omega L = j\, 10\text{k}\; 200\text{mH} = j2\text{k}\Omega$

$$z_C = \frac{-j}{\omega C} = \frac{-j}{10\text{k}\; 12.5\text{nF}} = \frac{-j}{125\,\mu}\,\Omega$$

$$z_C = \frac{-j\; 8\text{k}\,\Omega}{8\text{k} \cdot 125\,\mu} = -j8\text{k}\Omega$$



c) The value for $\mathbb{I}_L$ is given by the current divider formula:

$$\mathbb{I}_L = \mathbb{I}_s \cdot \frac{R \parallel Z_C}{R \parallel Z_C + Z_L}$$

$$= \mathbb{I}_s \frac{1}{1 + \dfrac{Z_L}{R \parallel Z_C}}$$

$$= \mathbb{I}_s \frac{1}{1 + Z_L\left(\dfrac{1}{R} + \dfrac{1}{Z_C}\right)}$$

$$= 25\angle 0^\circ \text{ mA} \frac{1}{1 + j2k\Omega\left(\dfrac{1}{2k\Omega} + \dfrac{1}{-j8k\Omega}\right)}$$

$$= 25\angle 0^\circ \text{ mA} \frac{4}{4} \frac{1}{1 + j - \frac{1}{4}}$$

$$= 25\angle 0^\circ \text{ mA} \frac{4}{3 + j4}$$

$$= 25\angle 0^\circ \text{ mA} \frac{4}{3 + j4} \cdot \frac{3 - j4}{3 - j4}$$

$$= \cancel{25}\angle 0^\circ \text{ mA} \frac{12 - j16}{\cancel{3^2 + 4^2}}$$

$$= 1\angle 0^\circ \text{ mA} \cdot 20\angle -53.1^\circ$$

$$\mathbb{I}_L = 20 \text{ mA} \angle -53.1^\circ$$

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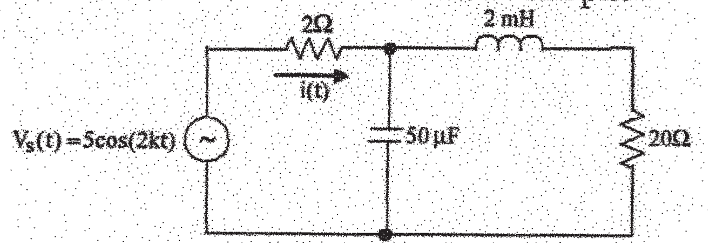
Homework #7 Examples

3.

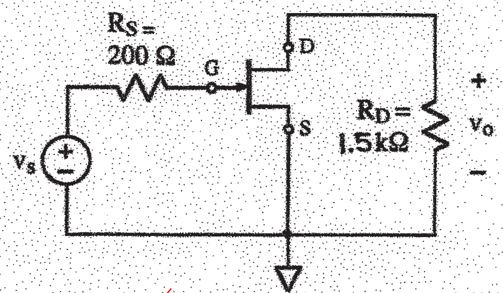


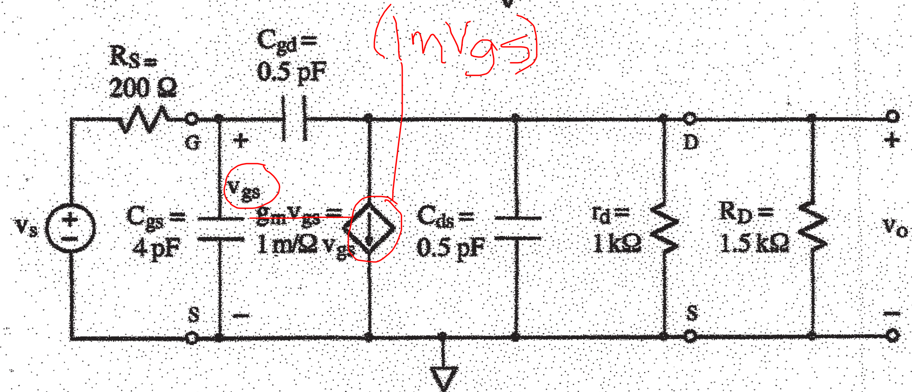
a. Find the phasor value for $V_s(t)$.

b. Draw the frequency-domain circuit diagram, including the phasor value for $V_s(t)$ and impedance values for components.

b. Find the phasor value for i(t).

Ex:





$v_s(t) = 2\cos(10kt)$ V

The above circuit diagrams show a common-source JFET amplifier and its high-frequency equivalent circuit. Find $v_o(t)$.

Sol'n: In this practical circuit, we have circuit values that allow us to make simplifying approximations.

We first calculate impedance values.

$\omega = 10k$ r/s from $v_s(t) = 2\cos(10kt)$ V

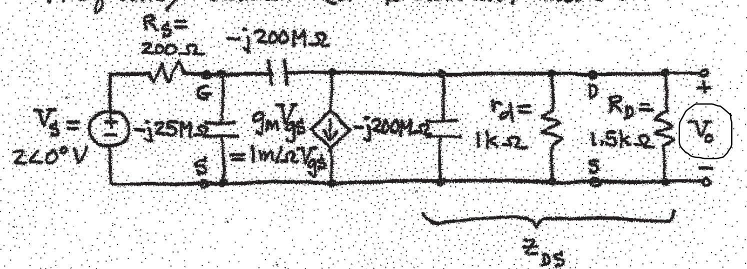
$$z_{C_{gs}} = \frac{-j}{\omega C_{gs}} = \frac{-j\ \Omega}{10k \cdot 4p} = -j\,25\text{M}\,\Omega$$

$$z_{C_{gd}} = \frac{-j}{\omega C_{gd}} = \frac{-j}{10k\,\frac{1}{2}\,p} = -j\,200\text{M}\,\Omega$$

$$z_{C_{ds}} = \frac{-j}{\omega C_{ds}} = \frac{-j}{10k\,\frac{1}{2}\,p} = -j\,200\text{M}\,\Omega$$

The phasor for $v_s(t)$ is $V_s = 2\angle 0°\ \text{V}$.

Frequency domain (or s-domain) model:



$$z_{DS} = -j\,200\text{M} \,\|\, 1\text{k}\Omega \,\|\, 1.5\text{k}\Omega$$

Starting with $1\text{k}\Omega \| 1.5\text{k}\Omega$ we have

$$1\text{k}\Omega \,\|\, 1.5\text{k}\Omega = 500\Omega \cdot 2\|3 = 500 \cdot \frac{6}{5}$$

$$= 600\,\Omega$$

Thus, $z_{DS} = -j\,200\text{M} \,\|\, 600\,\Omega = \dfrac{1}{\dfrac{1}{600} - \dfrac{1}{j200\text{M}}}$

Using $-\frac{1}{j} = j$ and rationalizing gives

$$z_{DS} = \frac{1}{\frac{1}{600} + \frac{j}{200M}} \cdot \frac{\frac{1}{600} - \frac{j}{200M}}{\frac{1}{600} - \frac{j}{200M}} \; \Omega$$

$$= \frac{\frac{1}{600} - \frac{j}{200M}}{\left(\frac{1}{600}\right)^2 + \left(\frac{1}{200M}\right)^2} \; \Omega$$

$$\approx \frac{\frac{1}{600} - \frac{j}{200M}}{\left(\frac{1}{600}\right)^2} \; \Omega \quad \text{Since } \frac{1}{200M} \ll \frac{1}{600}$$

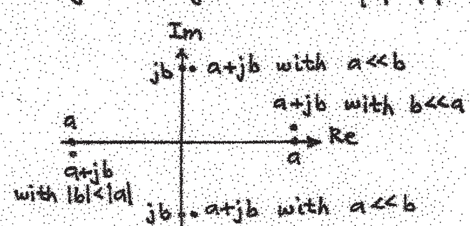
$$z_{DS} \approx \frac{\frac{1}{600}}{\left(\frac{1}{600}\right)^2} \; \Omega = \underline{600 \, \Omega}$$

In retrospect, we could have made the approximation that $-j200M \| 600\,\Omega \approx 600\,\Omega$.

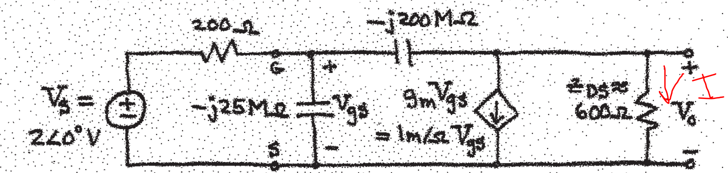
We may make this approximation despite the j in one of the quantities. In general, we may make the following approximations of complex values:

$$a + jb \approx a \quad \text{when } |b| \ll |a|$$

$$a + jb \approx jb \quad \text{when } |a| \ll |b|$$

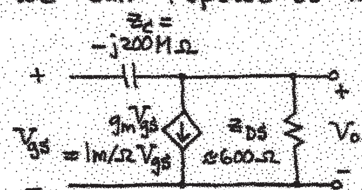


With our z_{DS} value, we have a simplified model:



$V_o = I(600)$

We now analyze the dependent source so we can replace it with an impedance, z_{eq}.



$$z_{eq} = \frac{V_o}{g_m V_{gs}}$$ using Ohm's law to write $z_{eq} = V/I$

Now we find a way to write V_o in terms of V_{gs}. We use a V-divider:

$$V_o = V_{gs} \frac{z_{eq} \| z_{DS}}{z_{eq} \| z_{DS} + z_C}$$

Substituting for V_o in our z_{eq} eq'n, we have

$$z_{eq} = \frac{\cancel{V_{gs}} \dfrac{z_{eq} \| z_{DS}}{z_{eq} \| z_{DS} + z_C}}{\cancel{V_{gs}} \cdot g_m}$$

$$z_{eq} = \frac{1}{g_m} \frac{\dfrac{z_{eq} z_{DS}}{z_{eq} + z_{DS}}}{\dfrac{z_{eq} z_{DS}}{z_{eq} + z_{DS}} + z_C}$$

$$z_{eq} = \frac{1}{g_m} \frac{z_{eq} z_{DS}}{z_{eq} z_{DS} + z_C (z_{eq} + z_{DS})}$$

Dividing top and bottom by z_{eq} gives the following:

$$z_{eq} = \frac{1}{g_m} \frac{z_{DS}}{z_{DS} + z_C + \frac{z_C z_{DS}}{z_{eq}}}$$

$$z_{eq}\left(z_{DS} + z_C + \frac{z_C z_{DS}}{z_{eq}}\right) = \frac{1}{g_m} z_{DS}$$

$$z_{eq}(z_{DS} + z_C) + z_C z_{DS} = \frac{1}{g_m} z_{DS}$$

$$z_{eq}\frac{(z_{DS} + z_C)}{z_{DS}} = \frac{\frac{1}{g_m}\cancel{z_{DS}} - z_C \cancel{z_{DS}}}{\cancel{z_{DS}}}$$

$$z_{eq} = \frac{\frac{1}{g_m} - z_C}{1 + \frac{z_C}{z_{DS}}} = \frac{\frac{1\,\Omega}{1\,\text{m}} - -j200\,\text{M}\Omega}{1 + \frac{-j200\,\text{M}\Omega}{600\,\Omega}}$$

$$z_{eq} = \frac{1\text{k} + j200\text{M}}{1 - j\frac{1}{3}\text{M}}\ \Omega$$

The imaginary parts of the numerator and denominator are much larger than the real parts. Thus, we ignore the real parts.

$$z_{eq} \approx j200\,\text{M}\Omega / -j\tfrac{1}{3}\text{M} \approx -600\,\Omega$$

Now we have a problem: $z_{eq} \| z_{DS} = -\frac{600^2}{0}\,\Omega$.

That means $z_{eq} \| z_{DS} = \infty\ \Omega$.

It is a good idea to try a more exact calculation to be sure that $z_{eq} \| z_{DS}$ is much larger than $z_C = -j200\,\text{M}\Omega$.

We use conductance to simplify calculations.

$$\frac{1}{z_{eq} \| z_{DS}} = \frac{1}{z_{eq}} + \frac{1}{z_{DS}} = \frac{1 + \frac{z_C}{z_{DS}}}{\frac{1}{g_m} - z_C} + \frac{1}{z_{DS}}$$

$$= \frac{1}{z_{DS}}\left(\frac{z_{DS} + z_C}{\frac{1}{g_m} - z_C} + 1\right)$$

$$= \frac{1}{z_{DS}}\left(\frac{z_{DS} + z_C}{\frac{1}{g_m} - z_C} + \frac{\frac{1}{g_m} - z_C}{\frac{1}{g_m} - z_C}\right)$$

$$= \frac{1}{z_{DS}}\left(\frac{z_{DS} + \frac{1}{g_m}}{\frac{1}{g_m} - z_C}\right)$$

$$= \frac{g_m + \frac{1}{z_{DS}}}{1 - g_m z_C}$$

$$= \frac{1\text{m} + \frac{1}{600}}{1 - 1\text{m}(-j200\text{M})}\ /\Omega$$

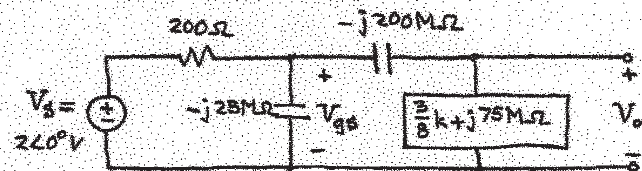
$$z_{eq} \| z_{DS} = \frac{1 + j200\text{k}}{\frac{1}{1\text{k}} + \frac{1}{600}}\ \Omega$$

$$z_{eq} \parallel z_{DS} = \frac{1 + j\,200k}{\frac{3+5}{3k}}\ \Omega = \frac{3}{8}k\,(1 + j200k)\ \Omega$$

$$= \frac{3}{8}k + j\,75M\,\Omega$$

We see that the value is <u>smaller</u> than $z_C = -j\,200\ M\Omega$.

Our new, simplified model:



We use V-dividers to find V_o.

$$V_{gs} = V_s \cdot \frac{-j25M\Omega \parallel \left(-j200M + \overset{\text{small}}{\cancel{\tfrac{3}{8}k}} + j\,75M\Omega\right)}{200 + -j25M\Omega \parallel \left(-j200M + \overset{\text{small}}{\cancel{\tfrac{3}{8}k}} + j75M\Omega\right)}$$

$$V_{gs} \doteq \frac{V_s \cdot (-j25M\Omega \parallel -j\,125\,M\Omega)}{200 - j25M\Omega \parallel -j125M\Omega}$$

where $-j25M\Omega \parallel -j125M\Omega = -j25M\Omega \cdot 1 \parallel 5$

$$= -j25M\Omega \cdot \frac{5}{6}$$

$$= -j\,\frac{125}{6}\ M\Omega$$

$$V_{gs} = \frac{V_s \cdot \left(-j\,\frac{125}{6}M\Omega\right)}{\overset{\text{small}}{\cancel{200}} - j\,\frac{125}{6}M\Omega} \approx V_s$$

$$V_o = V_{gs} \frac{\cancel{\tfrac{3}{8}k}^{\,\text{small}} + j\,75\,M\Omega}{\cancel{\tfrac{3}{8}k}^{\,\text{small}} + j75\,M\Omega - j\,200\,M\Omega}$$

$$V_o \approx V_{gs} \frac{j\,75\,M\Omega}{-j\,125\,M\Omega} = V_{gs}\left(-\frac{3}{5}\right)$$

$$V_o \approx 2\angle 0^\circ V \left(\frac{-3}{5}\right) = -\frac{6}{5}\angle 0^\circ V = \boxed{\frac{6}{5}\angle 180^\circ V}$$

Note: a minus sign is the same as 180° of phase shift.

$$v_o(t) = \frac{6}{5}\cos(10kt + 180^\circ)\ V$$

Homework #7 Examples

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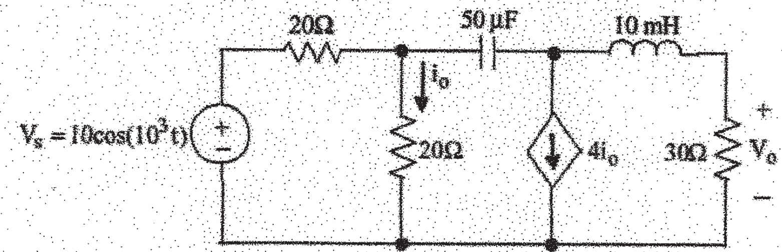
4.



Find $V_0(t)$.

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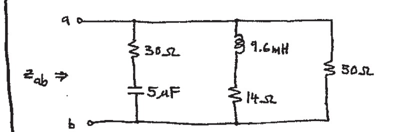
HW #7 Example

1.



$\omega = 5k\ r/s$

Find Z_{ab}.

sol'n: Convert C value to impedance $Z_C = -\frac{j}{\omega C}$.

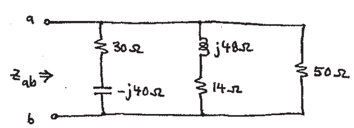
$$Z_C = \frac{-j}{5k\,r/s \cdot 5\,\mu F} = \frac{-j}{25m}\,\Omega = -j40\,\Omega$$

Convert L value to impedance $Z_L = j\omega L$.

$$Z_L = j\,5k\,r/s \cdot 9.6\,mH = j\,48\,\Omega$$

R values are the same as Z_R values.

Frequency-domain model:



$$Z_{ab} = (30-j40)\,\|\,(14+j48)\,\|\,50\ \Omega$$

For parallel Z's, it is often easir to use conductance, $\frac{1}{Z}$.

$$\frac{1}{Z_{ab}} = \frac{1}{30-j40\,\Omega} + \frac{1}{14+j48\,\Omega} + \frac{1}{50\,\Omega}$$

sol'n: 1. cont. Rationalize the $\frac{1}{Z}$ values and sum them.

$$\frac{1}{Z_{ab}} = \frac{30+j40}{30^2+40^2\,\Omega} + \frac{14-j48}{14^2+48^2\,\Omega} + \frac{1}{50\,\Omega}$$

$$\frac{1}{Z_{ab}} = \frac{30+j40}{50^2\,\Omega} + \frac{14-j48}{50^2\,\Omega} + \frac{50}{50^2\,\Omega}$$

Conveniently, we have the same denominator for every term. (Not typical!)

$$\frac{1}{Z_{ab}} = \frac{30+14+50 + j(40-48)}{50^2\,\Omega} = \frac{94-j8}{50^2\,\Omega}$$

$$Z_{ab} = \frac{50^2\,\Omega}{94-j8} = \frac{50^2\,(94+j8)}{(94-j8)(94+j8)} = \frac{50^2(94+j8)}{8900}$$

$$Z_{ab} = \frac{25}{89}\cdot(94+j8)\,\Omega \doteq 26.4 + j\,2.25\,\Omega$$

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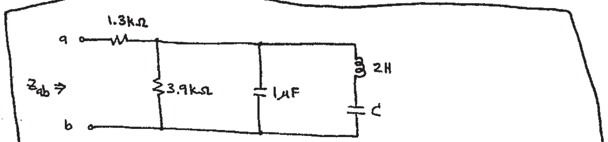
2.



Find value of C that makes Z_{ab} real. $\omega = 1\text{k r/s}$.

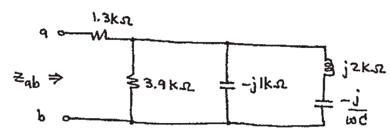
sol'n: First, convert C and L values to Z's.

$$Z_{1\mu F} = \frac{-j}{\omega C} = \frac{-j}{1\text{kr/s}\cdot 1\mu F} = \frac{-j\Omega}{1\text{m}} = -j1\text{k}\Omega$$

$$Z_{2H} = j\omega L = j\,1\text{kr/s}\cdot 2H = j2\text{k}\Omega$$

$$Z_C = \frac{-j}{\omega C}$$

Frequency domain model



$$Z_{ab} = 1.3\text{k}\Omega + 3.9\text{k}\Omega \,\|\, -j1\text{k}\Omega \,\|\left(j2\text{k}\Omega - \frac{j}{\omega C}\right)$$

If Z_{ab} is to be pure real, we must have that $3.9\text{k}\Omega \,\|\, -j1\text{k}\Omega \,\|\left(j2\text{k}\Omega - \frac{j}{\omega C}\right)$ is pure real. This follows from

$Z_{ab} - 1.3\text{k}\Omega$ = real # − real # = real #.

We can also show that real $\|$ Z = real if and only if Z is pure real (or ∞, see below)

real $\|$ Z = real ⇒ $\frac{1}{\text{real} \| Z} = \frac{1}{\text{real}}$ = real

⇒ $\frac{1}{\text{real}} + \frac{1}{Z}$ = real ⇒ $\frac{1}{Z}$ = real ⇒ Z = real or ∞

sol'n: 2. cont.

Thus, $-j1\text{k}\Omega \,\|\left(j2\text{k}\Omega - \frac{j}{\omega C}\right)$ = real

But this means $j\cdot -1\text{k}\Omega \,\|\left(2\text{k} - \frac{1}{\omega C}\right)$ = real.

Since the left side is of form j · real, the left side is pure imaginary unless it is zero.

$\therefore$ $-1\text{k}\Omega \,\|\left(2\text{k} - \frac{1}{\omega C}\right) = 0$

or $$\frac{-1\text{k}\Omega\cdot\left(2\text{k}\Omega - \frac{1}{\omega C}\right)}{-1\text{k}\Omega + 2\text{k}\Omega - \frac{1}{\omega C}} = 0$$

It follows that $2\text{k}\Omega - \frac{1}{\omega C} = 0$

or $\frac{1}{\omega C} = 2\text{k}\Omega$ or $\omega C = \frac{1}{2\text{k}\Omega}$

or $C = \frac{1}{\omega\cdot 2\text{k}\Omega}$ or $C = \frac{1}{1\text{k}\cdot 2\text{k}}F$

$\boxed{C = 0.5\,\mu F}$ $Z_{ab} = 1.3\text{k}\Omega$

Earlier, we noted that if $-1\text{k}\Omega \,\|\left(2\text{k}\Omega - \frac{1}{\omega C}\right) = \infty\,\Omega$ we also have a sol'n.

We achieve this if the denominator of the parallel impedance becomes zero.

$$-1\text{k}\Omega + 2\text{k}\Omega - \frac{1}{\omega C} = 0$$

or $\frac{1}{\omega C} = 1\text{k}\Omega$ or $\omega C = \frac{1}{1\text{k}\Omega}$

or $C = \frac{1}{\omega\cdot 1\text{k}\Omega} = \frac{1}{1\text{k}\cdot 1\text{k}\Omega} = 1\,\mu F$

$\boxed{C = 1\,\mu F}$ $Z_{ab} = 1.3\text{k}\Omega + 3.9\text{k}\Omega$

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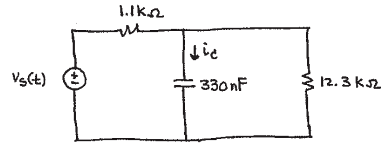
3.



$v_s(t) = 15\cos(3kt)$ V

a. Find phasor for $v_s(t)$.

b. Draw frequency domain circuit diagram.

c. Find phasor for $i_c(t)$.

d. Find $i_c(t)$.

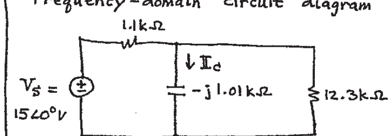
sol'n: a) $P[v_s(t)] = P[15\cos(3kt)\,V] = \boxed{15\angle 0°\,V \equiv V_s}$

b) From $v_s(t)$, $\omega = 3k$ r/s.

Impedance $Z_c = \frac{-j}{\omega C} = \frac{-j}{3k\ 330n}\ \Omega = \frac{-j}{990}\ M\Omega$

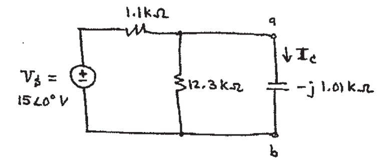
$Z_c = -j\,1.01\,k\Omega \approx -j\,1k\Omega$

Frequency-domain circuit diagram



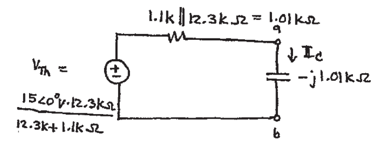
c) We could use the node-voltage method or mesh currents, but we can rearrange the circuit and use a Thevenin equivalent involving only R's. This reduces the number of calculations with complex quantities.

sol'n: 3.c) cont.



We take the Thevenin equivalent of the circuit to the left of the a,b terminals. Notice that the above circuit is the same as the original since the 12.3kΩ and −j 1.01 kΩ are still in parallel.

Using Thevenin equivalent, we have



$V_{Th} = \frac{15\angle 0° V \cdot 12.3k\Omega}{12.3k + 1.1k\Omega} \doteq 13.8\,V\angle 0°$

Now we can calculate I_c from V_{Th}/Z_{Tot}:

$$I_c = \frac{13.8\angle 0° V}{1.01k - j\,1.01k\Omega} = \frac{13.8\angle 0° V}{1.01k\Omega\,(1-j)}\,\frac{(1+j)}{(1+j)}$$

$$= \frac{13.8\angle 0° V}{1.01k\Omega \cdot 2}(1+j) = 6.83\,mA \cdot \sqrt{2}\angle 45°$$

$$\boxed{I_c = 9.66\angle 45°\ mA}$$

d) $i_c(t) = P^{-1}[I_c] = P^{-1}[9.66\angle 45°\ mA]$

$$\boxed{i_c(t) = 9.66\cos(3kt + 45°)\ mA}$$

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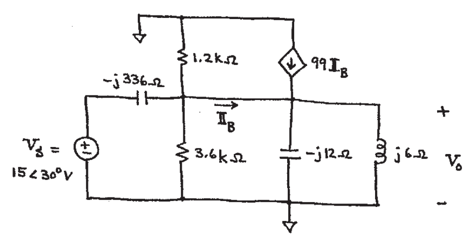
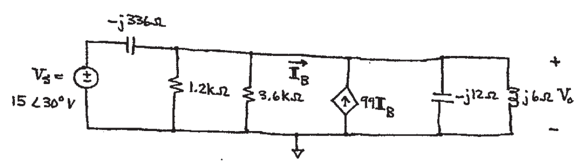
4.



$\omega = 10k$ r/s

Find $v_o(t)$.

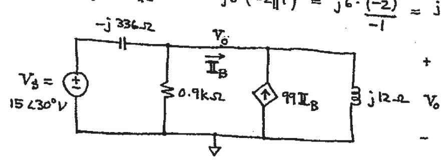
Sol'n: Redraw circuit, being careful to preserve I_B.



Combine parallel z's, again preserving I_B.

$1.2k\Omega \parallel 3.6k\Omega = 1.2k\Omega \cdot 1\parallel 3 = 12k\Omega \cdot \frac{3}{4} = 0.9k\Omega$

$-j12\Omega \parallel j6\Omega = j6(-2\parallel 1) = j6 \cdot \frac{(-2)}{-1} = j12\Omega$



If we try to use node-voltage to find V_o, we must express I_B in terms of V_o.

We use a current sum at the node above the $99I_B$ src.

Sol'n: 4. cont.

$$-I_B - 99I_B + \frac{V_o}{j12\Omega} = 0A$$

$$100\, I_B = \frac{V_o}{j12\Omega}$$

$$I_B = \frac{V_o}{j1.2k\Omega}$$

Now we can write node-voltage eq'n for V_o:

$$\frac{V_o - 15\angle 30°V}{-j336\Omega} + \frac{V_o}{0.9k\Omega} - \frac{99\,V_o}{j1.2k\Omega} + \frac{V_o}{j12\Omega} = 0A$$

$$V_o\left(\frac{1}{-j336\Omega} + \frac{1}{0.9k\Omega} - \frac{99}{j1.2k\Omega} + \frac{100}{j1.2k\Omega}\right) = \frac{-15\angle 30°V}{j336\Omega}$$

$$V_o\left(\frac{j}{336\Omega} + \frac{1}{0.9k\Omega} + \frac{-j}{1.2k\Omega}\right) = \frac{j15\angle 30°V}{336\Omega}$$

$$V_o = j15\angle 30°V\ \frac{(-j336\Omega \parallel 0.9k\Omega \parallel j1.2k\Omega)}{336\Omega}$$

$$-j336\Omega \parallel j1.2k\Omega = j12 \cdot 28 \parallel 100 = j48 \cdot -7\parallel 25\ \Omega$$

$$'' = j\cancel{48}^{8}\frac{(-175)}{18_{3}} = -j\frac{1400}{3}\Omega$$

$$0.9k\Omega \parallel -j\frac{1400}{3}\Omega = 100 \cdot 9 \parallel -j\frac{14}{3}\ \Omega$$

$$'' = 100\frac{(-j42)}{9 - j\frac{14}{3}} = 100\frac{(-j42)\cdot 3}{27 - j14}$$

$$'' = \frac{-j4(25)\,6(7)\cdot 3}{27-j14} = -j\frac{\overbrace{28 \cdot 12}^{336} \cdot 25 \cdot 3/2}{27-j14}$$

$$\therefore V_o = \frac{\cancel{j}15\angle 30°V\,(\cancel{j})\,25 \cdot 3/2}{27 - j14} \cdot \frac{27+j14}{27+j14}$$

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soln: 4. cont.

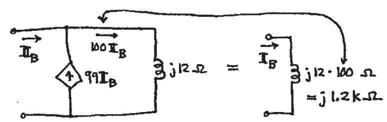
$$V_o = \frac{15\angle 30^\circ (\cancel{25})(3/2)}{\frac{925}{37}} \cdot \underbrace{(27+j14)}_{30.4\angle 27.4^\circ} \text{ V}$$

$$V_o \doteq \frac{15(3)\ 30.4}{2\cdot 37} \angle 30^\circ + 27.4^\circ \text{ V}$$

$$V_o \doteq 18.5 \angle 57.4^\circ \text{ V}$$

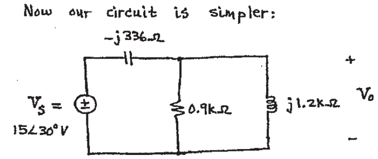
$$\boxed{v_o(t) \doteq 18.5 \cos(10kt + 57.4^\circ) \text{ V}}$$

An alternate approach is to use the idea of impedance multiplication.



The current thru the $j12\,\Omega$ causes a voltage drop equivalent to I_B flowing thru $j12\,\Omega \cdot 100$.

Now our circuit is simpler:



$$V_o = \frac{V_s \cdot 0.9k \| j1.2k\,\Omega}{(0.9k \| j1.2k\,\Omega) - j336\,\Omega}$$

$$0.9k \| j1.2k\,\Omega = 0.3k\,\Omega \cdot 3 \| j4 = 0.3k\,\Omega \cdot \frac{j12}{3+j4}$$

soln: 4. cont.

$$0.9k \| j1.2k\,\Omega = \overset{12}{\cancel{300}} \cdot \frac{j12}{\cancel{3^2+4^2}} (3-j4)\,\Omega = j144(3-j4)\,\Omega$$

$$'' = 576 + j432$$

$$V_o = \frac{15\angle 30^\circ \text{ V} \cdot (576+j432)\,\Omega}{576 + j432 - j336\,\Omega}$$

$$= \frac{15\angle 30^\circ \text{ V}\ (576+j432)}{576+j96}$$

$$= \frac{15\angle 30^\circ \text{ V} \cdot 720\angle 36.9^\circ}{584\angle 9.5^\circ}$$

$$= \frac{15(720)}{584} \angle 30^\circ + 36.9^\circ - 9.5^\circ \text{ V}$$

$$V_o = 18.5 \angle 57.4^\circ \text{ V}$$

$$\boxed{v_o(t) = 18.5 \cos(10kt + 57.4^\circ) \text{ V}}$$ as before