Frequency Response
In the Capacitors lab you made a "frequency dependent voltage divider" whose output was not the same for all frequencies of input. You made a graph of the output voltage as a function of the input frequency. That was a frequency response graph of the circuit. You made similar graphs in the Resonance lab. These graphs help show the relationship of the output to the input as a function of frequency. This relationship is known as the frequency response of the circuit. You may have heard the term used before in connection with speakers or microphones. All electrical and mechanical systems have frequency response characteristics. Sometimes the frequency response can be quite dramatic, like the Tacoma Narrows bridge.

Filter Circuits
A circuit which passes some frequencies and filters out other frequencies is called (surprise, surprise) a "filter" and this selection and rejection of frequencies is called "filtering". The tone or equalization controls on your stereo are frequency filters. So are the tuners in TVs and radios.

If a filter passes high frequencies and rejects low frequencies, then it is a high-pass filter. Conversely, if it passes low frequencies and rejects high ones, it is a low-pass filter. A filter that passes a range or band of frequencies and rejects frequencies lower or higher than that band, is a band-pass filter. The opposite of this is a band-rejection filter, or if the band is narrow, a notch filter or trap.

Look at the circuit at right. At low frequencies the impedance of the inductor is low and the output voltage is essentially shorted to ground. At high frequencies the impedance of the inductor is high and the output is about the same as the input. This is a high-pass filter. We can determine the relationship between the input and output:

\[ V_{\text{out}} = \frac{j \omega L}{R + j \omega L} V_{\text{in}} \]

OR:

\[ \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{j \omega L}{R + j \omega L} = H(\omega) \]

\[ Z_L = j \omega L \]

= The "Transfer Function"

A transfer function is a general term used for any linear system that has an input and an output. It is simply the ratio of output to input. The idea is that if you multiply the input by the transfer function, you get the output.

\[ H(\omega) = \frac{j \omega L}{R + j \omega L} \]

At low frequencies: \( R >> j \omega L \) and \( H(\omega) \approx \frac{j \omega L}{R} \) output is proportional to frequency

At high frequencies: \( R << j \omega L \) and \( H(\omega) \approx \frac{j \omega L}{j \omega L} = 1 \) output is about the same as the input.

Naturally, a plot of the transfer function verses frequency would be a handy thing. You've already made similar plots in the lab. It turns out that these plots are best done on a log-log scale. Unfortunately, they are actually plotted on a semilog scale using a special unit in the vertical axis called the decibel (dB) and the log is built into this dB unit. The dB unit doesn't really simplify things, but it is widely used and you'll need to know about it, so here goes.

Decibels
Your ears respond to sound logarithmically, both in frequency and in intensity. Musical octaves are in ratios of two. "A" in the middle octave is 220 Hz, in the next, 440 Hz, then 880 Hz, etc...

It takes about ten times as much power for you to sense one sound as twice as loud as another.

10x power \( \approx 2x \) loudness

A bel is such a 10x ratio of power.

Power ratio expressed in bels = \( \log \frac{P_2}{P_1} \) bels The bel is named for Alexander Graham Bell, who did original research in hearing.

It is a logarithmic expression of a unitless ratio (like the magnitude of \( H(\omega) \) or gain of an amplifier).

The bel unit is never actually used, instead we use the decibel (dB, \( 1/10 \)th of a bel).

Power ratio expressed in dB = \( 10 \cdot \log \frac{P_2}{P_1} \) dB

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dB are also used to express voltage and current ratios, which is related to power when squared. \[ P = \frac{V^2}{R} = I^2R \]

Voltage ratio expressed in dB = \[ 10 \cdot \log \left( \frac{V_2^2}{V_1^2} \right) \] dB  \[ = 20 \cdot \log \left( \frac{V_2}{V_1} \right) \] dB

Current ratio expressed in dB = \[ 20 \cdot \log \left( \frac{I_2}{I_1} \right) \] dB

These are the most common formulas used for dB

Some common ratios expressed as dB

\[ 20 \cdot \log \left( \frac{1}{\sqrt{2}} \right) = -3.01 \text{dB} \]
\[ 10 \frac{3}{20} = 0.708 \]
\[ 20 \cdot \log \left( \sqrt{2} \right) = 3.01 \text{dB} \]
\[ 10 \frac{3}{20} = 1.413 \]

\[ 20 \cdot \log \left( \frac{1}{2} \right) = -6.021 \text{dB} \]
\[ 10 \frac{6}{20} = 0.501 \]
\[ 20 \cdot \log (2) = 6.021 \text{dB} \]
\[ 10 \frac{6}{20} = 1.995 \]

\[ 20 \cdot \log \left( \frac{1}{10} \right) = -20 \text{dB} \]
\[ 10 \frac{20}{20} = 0.1 \]
\[ 20 \cdot \log (10) = 20 \text{dB} \]
\[ 10 \frac{20}{20} = 10 \]

\[ 20 \cdot \log \left( \frac{1}{100} \right) = -40 \text{dB} \]
\[ 10 \frac{40}{20} = 0.01 \]
\[ 20 \cdot \log (100) = 40 \text{dB} \]
\[ 10 \frac{40}{20} = 100 \]

Other dB-based units

You may have encountered dB as an absolute measure of sound intensity (Sound Pressure Level or SPL). In that case the RMS sound pressure is compared as a ratio to a reference of \( 2 \times 10^{-5} \) Pascals.

dBm is another absolute power scale expressed in dB. Powers are referenced to 1mW.

Volume Units (VU) are dBm with the added spec that the load resistor is 600Ω.

Bode Plots

Named after Hendrik W. Bode (bo-dee), bode plots are just frequency response curves made on semilog paper where the horizontal axis is frequency on a \( \log_{10} \) scale and the vertical axis is either dB or phase angle. The plots are nothing special, but the method that Bode came up with to make them quickly and easily is special. We aren’t going to bother with the phase-angle plots in this class, but since the bode method of making frequency plots is so simple it’s worth our time to see how it’s done.

Basically, these are the steps:

1. Find the transfer function.
2. Analyze the transfer function to find “corner frequencies” and use these to divide the frequency into ranges.
3. Simplify and approximate the magnitude of the transfer function in each of these ranges.
4. Draw a “straight-line approximation” of the frequency response curve.
5. Use a few memorized facts to draw the actual frequency response curve.

The best way to learn the method is by examples.

Ex. 1

\[
\begin{align*}
V_{\text{out}} &= \frac{1}{j\omega C + R} = \frac{1}{1 + R \cdot (j\omega C)} = H(\omega) = \text{The “Transfer Function”} \\
\end{align*}
\]

corner frequency is where real = imaginary (in denominator in this case)

\[ 1 = \omega C \cdot R \cdot C \]
\[ \omega C := \frac{1}{R \cdot C} \]
\[ \omega C = 250 \frac{\text{rad}}{\text{sec}} \]

So...

\[ H(\omega) := \frac{1}{1 + j \frac{\omega}{250 \frac{\text{rad}}{\text{sec}}}} \]

\( \omega C \) is also called a "pole" frequency

The transfer function is said to have one "pole" at \( \omega C \)
To make a straight-line approximation of the magnitude of \( H(\omega) \) we’ll approximate \(|H(\omega)|\) in two regions, one below the corner frequency, and one above the corner frequency. Keep only the real or only the imaginary part of the denominator, depending on which is greater.

below the corner frequency: \( \omega < \omega_c \)

\[
H(\omega) \simeq \frac{1}{1 + \frac{\omega_c}{\omega}} \quad |H(\omega)| \simeq \frac{1}{1 + \log(10)} = 1 \quad 20 \log(1) = 0 \text{ dB}
\]

above the corner frequency: \( \omega > \omega_c \)

\[
H(\omega) \simeq \frac{1}{j \omega} \quad |H(\omega)| \simeq \frac{1}{\omega} \left( \frac{250 \text{ rad}}{\text{sec}} \right) \quad \text{inversely proportional to } \omega.
\]

Inverse proportionality is a straight 1 to 1 down slope on a log-log plot, with dB it’s a only slightly different. Since 10x corresponds to 20 dB, the line goes down 20 dB for every 10x increase in frequency (called a decade).

That’s all you need to make the straight-line approximation shown in the plot below. (If you know the slope)

Try some values above the corner frequency:

\[
20 \log \left[ \frac{1}{10 \cdot \omega_c} \left( \frac{250 \text{ rad}}{\text{sec}} \right) \right] = -20 \text{ dB}
\]

\[
20 \log \left[ \frac{1}{100 \cdot \omega_c} \left( \frac{250 \text{ rad}}{\text{sec}} \right) \right] = -40 \text{ dB}
\]

The slope above the corner frequency is -20 dB per “decade”.

A decade is a 10x increase in frequency.

This slope is also -6dB per “octave” (a 2x increase in frequency).

Let’s find the actual magnitude of \( H(\omega) \) right at the corner frequency \( (H(\omega_c)) \):

\[
\omega = \omega_c \quad H(\omega) = \frac{1}{1 + \frac{\omega_c}{\omega}} = \frac{1}{1 + j \cdot 1}
\]

\[
|H(\omega)| = \frac{1}{\sqrt{2}} \quad 20 \log \left( \frac{1}{\sqrt{2}} \right) = -3.01 \text{ dB}
\]
Ex. 2

\[ V_R \over V_S = \frac{50 - R}{1 + j\cdot\omega\cdot C} + R = \frac{50 \cdot (R - (j\cdot\omega\cdot C))}{1 + R \cdot (j\cdot\omega\cdot C)} = H(\omega) \]

Transfer function has one pole at \( \omega_c \)

corner frequency is where real = imaginary

\[ 1 = \frac{\omega_c}{R\cdot C} \quad \omega_c := \frac{1}{R\cdot C} \quad \omega_c = 500 \cdot \text{rad/sec} \]

So...

\[ H(\omega) := \frac{50 \cdot j\cdot\omega}{\omega_c} \quad \frac{50 \cdot j\cdot\omega}{\frac{50\cdot\text{rad}}{\text{sec}} + j\cdot\omega} \]

OR:

\[ H(\omega) := \frac{50 \cdot j\cdot\omega}{\omega_c} \quad \frac{50 \cdot j\cdot\omega}{1 + j\cdot\omega} \]

Proportional to \( \omega \). That's all we need to know here. This proportionality to \( \omega \) will result in a +20 dB per decade slope for all frequencies below the corner frequency

\[ \omega < \omega_c \quad H(\omega) \simeq \frac{50\cdot j\cdot\omega}{500 \cdot \text{rad/sec}} \cdot \frac{0.1 \cdot \sec}{\text{rad}} \]

\[ |H(\omega)| \simeq 0.1 \cdot \frac{\sec}{\text{rad}} \]

Actual value at the corner frequency

\[ \omega = \omega_c \quad H(\omega) = \frac{50\cdot j\cdot\omega}{500 \cdot \text{rad/sec} + j\cdot\omega} = \frac{50\cdot j\cdot\omega}{1 + j\cdot\omega} = 25 + 25j |25 + 25j| = 35.355 \quad 20 \cdot \log(35.355) = 30.97 \cdot \text{dB} \]

3 dB lower than the magnitude in the pass band

Magnitude plot

[Graph showing Bode plot with magnitude in dB plotted against frequency in radians per second.]

Straight-line approximation __________

Actual _______
Ex. 3 The transfer function may already be worked out: \[ H(f) := \frac{1 + j \frac{f}{10 \text{ Hz}}}{1 + j \frac{f}{500 \text{ Hz}}} \]

Could come from a circuit like this:

The real and imaginary parts of the numerator are equal at the one corner frequency (called a "zero")

\[ 1 = j \frac{f_c}{10 \text{ Hz}} \quad f_c1 := 10 \text{ Hz} \]

The real and imaginary parts of the denominator are equal at the other corner frequency (pole)

\[ 1 = j \frac{f_c}{500 \text{ Hz}} \quad f_c2 := 500 \text{ Hz} \]

There are now three regions to approximate \(|H(f)|\)

Below the first corner frequency: \( f < 10 \text{ Hz} \)

\[ |H(f)| \approx \left| 10 \frac{1}{1} \right| = 10 \quad 20 \cdot \log(10) = 20 \text{ dB} \]

Between the corner frequencies: \( 10 \text{ Hz} < f < 500 \text{ Hz} \)

\[ |H(f)| \approx \left| 10 \frac{j \frac{f}{10}}{1} \right| = f \quad \text{proportional to } f \]

Above the second corner frequency: \( 1000 \text{ Hz} < f \)

\[ |H(f)| \approx \left| 10 \frac{j \frac{f}{500}}{1} \right| = 500 \quad 20 \cdot \log(500) = 53.98 \text{ dB} \]
Ex. 4

A Transfer function of a typical amplifier:

$$H(\omega) := \frac{j\omega - 0.182 \text{sec}}{1 + \frac{j\omega}{6.875 \times 10^4 \text{rad/sec}}} \left(1 + \frac{j\omega}{416.67 \text{ rad/sec}}\right)$$

$$\omega_{C1} := 416.67 \frac{\text{rad}}{\text{sec}}$$

$$\omega_{C2} := 6.875 \times 10^4 \frac{\text{rad}}{\text{sec}}$$

Between the two poles (passband):

$$H(\omega) \simeq \frac{j\omega}{1 + \frac{j\omega}{416.67}} = 75.834$$

$$20 \log(75.834) = 37.6$$

Below $$\omega_{C1}$$

$$H(\omega) \simeq \frac{j\omega - 0.182}{(1)(j\omega)}$$ proportional to $$\omega$$

Above $$\omega_{C2}$$

$$H(\omega) \simeq \frac{j\omega - 0.182}{\left(\frac{j\omega}{6.875 \times 10^4}\right)\left(\frac{j\omega}{416.67}\right)}$$ inversely proportional to $$\omega$$

### Warning

The Bode plots that we’ve covered here are the simplest types and only magnitude plots. This will do for an initial introduction to simple filters, but this coverage is not complete.

Complete Bode plots also include phase plots which we haven’t looked at at all. Also, if some poles and zeroes are too close to each other they can interact and even result in complex poles.

If asked in a future classes if you have “covered” Bode plots, do not make the mistake of saying “yes”.
1.6 Second-Order Transients

A circuit with both a capacitor and an inductor is like a mechanical system with both a mass and a spring. When there are two different types of energy-storage elements, the transient responses can be much more interesting than the simple exponential curves that we've seen so far. Many of these systems can oscillate or "ring" when a transient is applied. When you analyze a circuit with a capacitor and an inductor you get a second-order differential equation, so the transient voltages and currents are called second-order transients.

**Series RLC circuit, traditional way:** Look at the circuit at right. The same current flows through all three elements ($i(t)$ or just $i$). That current will begin to flow after time $t = 0$, when the switch is closed. Using basic circuit laws:

$$V_{in} = v_R + v_L + v_C$$

$$= iR + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt$$

Making the obvious substitutions.

The next step here would be to differentiate both sides of the equation, but we've been through this before with the RC circuit. If you're a little more clever, there's an easier way.

Make this substitution instead $i = i_C = C\frac{d}{dt}v_C$, to get $V_{in} = R\frac{d}{dt}v_C + L\frac{d^2}{dt^2}v_C + v_C$

Rearrange this equation to get $V_{in} = L\frac{d^2}{dt^2}v_C + R\frac{d}{dt}v_C + v_C$ and $\frac{V_{in}}{L} = \frac{d^2}{dt^2}v_C + \frac{R}{L}\frac{d}{dt}v_C + \frac{1}{L}\frac{d}{dt}v_C$

This is the classical second-order differential equation and it is solved just like the first-order differential equation, by guessing a solution of the right form and then finding the particulars of that solution.

**Standard differential equation answer:** $v_C(t) = A + B\cdot e^{st}$

Note: It will turn out that there will be two $s$'s ($s_1$ and $s_2$), and two $B$'s ($B$ and $D$) for the second-order solution. For now I'll leave out that added complexity.

Differentiate: $\frac{d}{dt}v_C = B\cdot s\cdot e^{st}$

And again: $\frac{d^2}{dt^2}v_C = B\cdot s^2\cdot e^{st}$

Substitute these back into the original equation:

$$\frac{V_{in}}{L} = \frac{d^2}{dt^2}v_C + \frac{R}{L}\frac{d}{dt}v_C + \frac{1}{L}\frac{d}{dt}v_C$$

$$= B\cdot s^2\cdot e^{st} + \frac{R}{L}B\cdot s\cdot e^{st} + \frac{1}{L}\left(A + B\cdot e^{st}\right)$$

$$= B\cdot s^2\cdot e^{st} + \frac{R}{L}B\cdot s\cdot e^{st} + \frac{1}{L}\cdot\frac{d}{dt}v_C + \frac{1}{L}\cdot A$$

We can separate this equation into two parts, one which is time dependent and one which is not. Each part must still be an equation.

**Time independent (forced) part:** $V_{in} = A$, $A = V_{in} = \text{final condition} = v_C(\infty)$ just like before

**Time dependent (transient) part:** $0 = B\cdot s^2\cdot e^{st} + \frac{R}{L}B\cdot s\cdot e^{st} + \frac{1}{L}\cdot B\cdot e^{st}$

Divide both sides by $B\cdot e^{st}$ to get: $0 = s^2 + \frac{R}{L}s + \frac{1}{L}\cdot A = \text{characteristic equation}$

This equation is important. It is called the characteristic equation and we'll need to find one like it for every second-order circuit that we analyze. Luckily, there's a much easier way to get it, using impedances similar to those we used in phasor analysis. I'll talk about that in the next section, in the meantime, let's continue with this problem.

Transients p. 1.9
Once you have the characteristic equation:

\[ s^2 + \frac{R}{L}s + \frac{1}{L/C} = 0 \]

The characteristic equation is solved using the quadratic equation, recall:

if \( a \cdot x^2 + b \cdot x + c = 0 \)

there are two solutions

\[ x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

and

\[ x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

This results in three possible types of solutions, depending on what's under the radical, +, -, or 0.

Notice also that there are two s values \((s_1 \text{ and } s_2)\) and that leads to two two \(B\)'s (we'll call them \(B\) and \(D\)).

**Overdamped** The part under the radical is +

if \( \left(\frac{R}{L}\right)^2 - \frac{4}{L/C} > 0 \) then \( s_1 \) and \( s_2 \) are both real and \( s_1 \neq s_2 \) and our guessed solution \( v_C(t) = A + B \cdot e^{s_1 t} \)

will become \( v_C(t) = v_C(\infty) + B \cdot e^{s_1 t} + D \cdot e^{s_2 t} \) and is simply the combination of two exponentials.

Also both \( s_1 \) and \( s_2 \) will always be negative (unless you find a negative \( R, C, \) or \( L)\), meaning the exponential parts will decay with time and are thus transient.

This is the overdamped case, like a class of students on a Monday morning. Pretty dull and soon to be asleep.

**Underdamped** The part under the radical is -

if \( \left(\frac{R}{L}\right)^2 - \frac{4}{L/C} < 0 \) then \( s_1 \) and \( s_2 \) are both complex and can be expressed as

\[ s_1 = \alpha + j\omega \quad \text{and} \quad s_2 = \alpha - j\omega \]

Well, if you start putting complex numbers in exponentials, what do you get? Euler's equations show that you'll get sines and cosines. In this case it's much easier to rephrase the guessed solution like this.

\[ v_C(t) = v_C(\infty) + e^{\alpha t} \left( B_2 \cos(\omega t) + D_2 \sin(\omega t) \right) \]

This form can be derived directly from \( v_C(t) = A + B \cdot e^{s_1 t} + D \cdot e^{s_2 t} \)

using Euler's equation, \( e^{j\theta} = \cos(\theta) + j \sin(\theta) \), but we won't bother to here.

In fact, although \( B_2 \) and \( D_2 \) are not the same as \( B \) and \( D \), I'll drop the "2" subscripts because we'll never actually need to convert between these two forms and the extra subscripts just become annoying.

So: \( v_C(t) = v_C(\infty) + e^{\alpha t} \left( B \cos(\omega t) + D \sin(\omega t) \right) \)

\( \alpha \) and \( \omega \) come from the \( s_1 \) and \( s_2 \) solutions to the characteristic equation. \( \omega \) is frequency at which the underdamped circuit will "ring" or "oscillate" in response to a transient. \( \alpha \) sets the decay rate of that oscillation.

Because \( \alpha \) will always be negative the \( e^{\alpha t} \) term insures that the transient ringing dies out in time.

This is the underdamped case, like students on spring break in Fort Lauderdale.

**Natural Frequency and the Damping Ratio**

These are commonly used terms to describe the underdamped response in a normalized way, similar to the \( \tau \) used to describe first-order transient responses.

The "natural frequency" is defined as: \( \omega_n = \sqrt{\frac{\omega^2}{\alpha^2 - \omega^2}} \)

It is the frequency that the system would oscillate at if there were no damping \((R=0 \text{ in our case})\) for this case: \( \omega_n = \frac{1}{\sqrt{L/C}} \)

Transients p. 1.10
Critically damped  
The part under the radical is $0$ if $\frac{R^2}{L} \cdot \frac{4}{LC} = 0$ then $s_1$ and $s_2$ are both real and exactly the same. Now our guessed solution must be modified to $v_C(t) = v_C(\infty) + B \cdot e^{s_1 t} + D \cdot e^{s_2 t}$ and can result in a single overshoot.

This is actually a trivial case since it relies on an exact equality which will never happen in reality. The best use of the critically damped case is as a conceptual border between the over- and under-damped cases.

**RLC examples**

Let's use some component values in the RLC circuit and see what happens.

**Overdamped Example**

$$\frac{R^2}{L} \cdot \frac{4}{LC} > 0 \quad s_1 \text{ and } s_2 \text{ are real and negative, overdamped.}$$

$$s_1 := \frac{-R}{2L} + \frac{1}{2} \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \quad s_1 = -2000 \text{ sec}^{-1}$$

$$s_2 := \frac{-R}{2L} - \frac{1}{2} \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \quad s_2 = -2500 \text{ sec}^{-1}$$

$$v_C(t) = v_C(\infty) + B \cdot e^{s_1 t} + D \cdot e^{s_2 t}$$

(As an example, the form is the same for all variables in this circuit)

**Final conditions**

REDRAW the circuit with the inductor as a short and the capacitor as an open.

$$V_{in} = 12 \cdot V \quad i_L(\infty) = 0$$

$$v_C(\infty) = \text{final condition} = 12 \cdot V$$

The capacitor will eventually charge up to $V_{in}$.

**Initial conditions**

REDRAW the circuit before the switch changes. Find two initial conditions, $i_L(0^-)$ and $v_C(0^-)$.

$$v_R(0-) = i(0-)R = 0$$

$$i_L(0-) = 0 = i_L(0+)$$

$i_L$ and $v_C$ cannot change instantly, so cannot change the instant the switch changes.

$$v_C(0-) = 0 = v_C(0+)$$

Pretty easy in this case

(assuming no initial charge)

Transients p. 1.11
REDRAW the circuit again just after the switch changes. Show the inductor as a current source of $i_L(0)$ (same as $i_L(0-)$) and the capacitor as a voltage source of $v_C(0)$ (same as $v_C(0-)$). Find two more initial conditions, $v_L(0)$ and $i_C(0)$. Both $v_L(0)$ or $i_C(0)$ can change instantly, so you must find them from $i_L(0)$ and $v_C(0)$.

$$v_R(0) = i_L(0)R = 0V$$

$$+$$

$$-$$

$$+$$

$$-$$

$$v_L(0) = 12V$$

$$v_C(0) = 0$$

At time $t = 0$:

$$v_C(0) = v_C(\infty) + B + D = 0$$

Rearrange the basic equations for inductors and capacitors to find the initial slopes from $v_L(0)$ or $i_C(0)$.

Rearrange $v_L = L \frac{d}{dt} i_L$ to $\frac{d}{dt} i_L(0) = \frac{v_L(0)}{L} = \frac{12V}{600\cdot A} = 0.0\cdot A$ in this case

or, $i_C = C \frac{d}{dt} v_C$ to $\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C} = \frac{0.0V}{sec}$

Note: You will need only the first one if you are looking for $i_L(t)$.

You will need only the second one if you are looking for $v_C(t)$.

You may need both if you are looking for any other variable in the circuit.

Other variables can usually be found most easily from $i_L(t)$ and/or $v_C(t)$.

**To Find $v_C(t)$**

At time $t = 0$:

$$v_C(0) = v_C(\infty) + B + D = 0$$

This equation has two unknowns. The initial slope will give us the needed second equation.

$$0 = 12\cdot V + B + D$$

Rearranging: $D = -12\cdot V - B$

Differentiate the solution:

$$v_C(t) = v_C(\infty) + B \cdot e^{s\cdot t} + D \cdot e^{s\cdot 2\cdot t}$$

To get:

$$\frac{d}{dt} v_C(t) = 0 + B \cdot s \cdot e^{s\cdot t} + D \cdot 2\cdot s \cdot e^{s\cdot 2\cdot t}$$

At time $t = 0$:

$$\frac{d}{dt} v_C(0) = B \cdot s + D \cdot s$$

From initial conditions, above:

$$\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C} = 0.0\cdot V$$

Combining:

$$0.0\cdot \frac{V}{sec} = B \cdot s + D \cdot s$$

The second equation!

Solve simultaneously for $B$ and $D$:

$$0.0\cdot \frac{V}{sec} = B \cdot s + (-12\cdot V - B) \cdot s$$

$$B = \frac{12\cdot V}{s \cdot (s - 1)} = -60\cdot V$$

$D = -12\cdot V - B = -12\cdot V - 60\cdot V = 48\cdot V$
recall the solution: \( v_C(t) = v_C(\infty) + B \cdot e^{s \cdot t} + D \cdot e^{s \cdot 2t} \)

Substitute everything back in back in: \( v_C(t) := 12 \cdot V - 60 \cdot V \cdot e^{\frac{2000}{t} \cdot \text{sec}} + 48 \cdot V \cdot e^{\frac{2500}{t} \cdot \text{sec}} \)

Notice that this is not a simple exponential curve, although admittedly it's not much more interesting.

To Find i_L(t) or i_R(t) or i_C(t) which all the same i(t).

\( i(t) = i(\infty) + B \cdot e^{s \cdot t} + D \cdot e^{s \cdot 2t} \)

From final and initial conditions

\[
\begin{align*}
    i(0) &= i(\infty) + B + D = 0 = 0 + B + D & D = -B \\
    \frac{d}{dt}i(0) &= B \cdot s \cdot 1 + D \cdot s \cdot 2 = \frac{12 \cdot V}{L} = 600 \cdot \frac{A}{\text{sec}}
\end{align*}
\]

Solve simultaneously for B and D

\[
\begin{align*}
    \frac{12 \cdot V}{L} &= B \cdot s \cdot 1 - B \cdot s \cdot 2 \\
    B &= \frac{\left(\frac{12 \cdot V}{L}\right)}{s \cdot (1 - s \cdot 2)} = 1.2 \cdot A \\
    D &= -B = -1.2 \cdot A
\end{align*}
\]

Substitute back in: \( i(t) := 1.2 \cdot e^{\frac{2000}{\text{sec}} \cdot t} - 1.2 \cdot e^{\frac{2500}{\text{sec}} \cdot t} \cdot A \)

However you get to it, at least this curve is slightly more interesting than the \( v_C(t) \).

We could have found the same result from \( v_C(t) \), using that to find \( i_L(t) \):

\[
\begin{align*}
    i_C(t) &= C \cdot \frac{d}{dt}v_C(t) = C \cdot \frac{d}{dt} \left[ 12 \cdot V - 60 \cdot V \cdot e^{\frac{2000}{t} \cdot \text{sec}} + 48 \cdot V \cdot e^{\frac{2500}{t} \cdot \text{sec}} \right] \\
    &= C \cdot (-60 \cdot V) \cdot \left( \frac{2000}{\text{sec}} \right) \cdot e^{\frac{2000}{t} \cdot \text{sec}} + C \cdot 48 \cdot V \cdot \left( \frac{25}{\text{sec}} \right) \cdot e^{\frac{2500}{t} \cdot \text{sec}} \\
    C \cdot (-60 \cdot V) \cdot \left( \frac{2000}{\text{sec}} \right) &= 1.2 \cdot A \\
    C \cdot 48 \cdot V \cdot \left( \frac{2500}{\text{sec}} \right) &= -1.2 \cdot A \\
    i(t) &:= 1.2 \cdot e^{\frac{2000}{t} \cdot \text{sec}} - 1.2 \cdot e^{\frac{2500}{t} \cdot \text{sec}} \cdot \text{A}
\end{align*}
\]
**Underdamped Example**

\[ R := 10\, \Omega \quad L := 20\, \text{mH} \quad C := 10\, \mu \text{F} \]

\[
\begin{align*}
\sigma_1 &= \frac{R}{2L} + \frac{1}{2} \sqrt{\frac{R^2}{L^2} - \frac{4}{L/C}} \\
\sigma_2 &= \frac{R}{2L} - \frac{1}{2} \sqrt{\frac{R^2}{L^2} - \frac{4}{L/C}}
\end{align*}
\]

\[ s_1 = -250 + 2.222 \times 10^3 \text{j sec}^{-1} \]

\[ s_2 = -250 - 2.222 \times 10^3 \text{j sec}^{-1} \]

\[ \alpha := \frac{-250}{\text{sec}} \]

\[ \omega := \text{Im}(s_1) \quad \omega = 2222 \frac{\text{rad}}{\text{sec}} \]

The final and initial conditions are the same as before, since they did not depend on \( R \) and \( R \) is the only component that is different.

Let's find the current again this time.

\[ i(t) = i(\infty) + e^{\sigma t} \left( B \cos(\omega t) + D \sin(\omega t) \right) \quad \text{(underdamped this time)} \]

\[ i(0) = i(\infty) + B, \]

\[ 0 = 0 + B \quad B := 0 \text{A} \]

Differentiate the solution: \( i(t) = i(\infty) + e^{\sigma t} \left( B \cos(\omega t) + D \sin(\omega t) \right) \)

\[ \frac{d}{dt} i(t) = \alpha e^{\sigma t} \left( B \cos(\omega t) + D \sin(\omega t) \right) + e^{\sigma t} \left( -B \omega \sin(\omega t) + D \omega \cos(\omega t) \right) \]

At time \( t = 0 \):

\[ \frac{d}{dt} i(0) = B \alpha + D \omega \]

Solve for \( D \): \[ D = \frac{\frac{d}{dt} i(0) - B \alpha}{\omega} \]

\[ D = \frac{12 \text{V}}{L} \]

\[ D = \frac{12 \text{V} - B \alpha}{\omega} = 0.27 \text{A} \]

Substitute back in: \[ i(t) = e^{\sigma t} \left( 0.27 \sin(\omega t) \right) \]

Now this is much more interesting.
Critically Damped Example

First we have to figure out how to get this case

Change R’s value to create critical damping:
\[
\frac{\left(\frac{R}{L}\right)^2 - \frac{4}{L\cdot C}}{\sqrt{\frac{R}{L}} - \frac{4}{L\cdot C}} = 0 \Rightarrow R = 89.44271909999159 \cdot \Omega
\]
(exactly)

\[s_1 = -\frac{R}{2L} + \frac{1}{2} \sqrt{\frac{R}{L}} - \frac{4}{L\cdot C} = -2236 \cdot \text{sec}^{-1}\]
\[s_2 = -\frac{R}{2L} - \frac{1}{2} \sqrt{\frac{R}{L}} - \frac{4}{L\cdot C} = -2236 \cdot \text{sec}^{-1}\]

\[i(t) = i(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot t \cdot e^{s_2 \cdot t}\]
\[i(\infty) = \text{final condition} = 0 \cdot \text{A} \quad \text{Capacitor will charge up and current will stop.}\]

\[i(0) = i(\infty) + B = 0, \quad B = 0\]

\[\frac{d}{dt} i(0) = B \cdot s_1 \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t} + D \cdot t \cdot s_2 \cdot e^{s_2 \cdot t} = B \cdot s_1 + D = \frac{12 \cdot V}{L}\]
Since all initial voltage will be across inductor.

Solve for D:
\[D = \frac{12 \cdot V}{L} = 600 \cdot \frac{A}{\text{sec}}\]

Substitute back in:
\[i(t) = 600 \cdot \frac{A}{\text{sec}} \cdot t \cdot e^{-2236 \cdot t}\]

if you notice a remarkable similarity with the overdamped case, that’s common for critical damping.
1.7 The Easy Way to get the Characteristic Equation

Recall from your Ordinary Differential Equations class, the Laplace transform method of solving differential equations. The Laplace transform allowed you to change time-domain functions to frequency-domain functions. We've already done this for steady-state AC circuits. We changed functions of \( t \) into functions of \( j\omega \). That was the frequency domain. Laplace let's us do the same sort of thing for transients. The general procedure is as follows.

1) Transform your forcing functions into the frequency domain with the Laplace transform.

2) Solve your differential equations with plain old algebra, where: \( \frac{d}{dt} \) operation can be replaced with \( s \), and \( \int dt \) can be replaced by \( \frac{1}{s} \).

3) Transform your result back to the time domain with the inverse Laplace transform.

Step 1 isn't too bad, but step 3 can be a total pain without a good computer program to do the job. However, step 2 sounds great. It turns out that we can use step 2 alone and still learn a great deal about our circuits and other systems without ever bothering with steps 1 and 3.

First remember from your study of Laplace that differentiation in the time domain was the same as multiplication by \( s \) in the frequency domain. That's really all we need and we're off and running.

\[
V_L(t) = L\frac{d}{dt}i_L(t) \quad \rightarrow \quad V_L(s) = LsI_L(s) \quad \text{and} \quad i_C(t) = C\frac{d}{dt}v_C(t) \quad \rightarrow \quad I_C(s) = CsV_C(s)
\]

Leading to the Laplace impedances: \( Ls \) for an inductor and \( \frac{1}{Cs} \) for a capacitor.

That's it, now we can use these impedances just like the \( j\omega \) impedances, and we can use all the tools developed for DC. And with Laplace we don't even have to mess with complex numbers.

Look what happens to the RLC circuit now.

Pick any dependent variable (\( I(s) \), \( V_R(s) \), \( V_L(s) \), or \( V_C(s) \)) and write a transfer function, which is a ratio of the dependent variable to the input (\( V_{in}(s) \)), like this:

\[
V_{in}(s) = I(s)\left(\frac{1}{Cs} + R + Ls\right)
\]

Transfer function \( = H(s) = \frac{I(s)}{V_{in}(s)} = \frac{1}{\left(\frac{1}{Cs} + R + Ls\right)} \)

Manipulate this transfer function into this form:

\[
f(a)s^2 + bs + k = \frac{1}{s^2 + b\cdot s + k}
\]

One polynomial divided by another.

\[
\frac{I(s)}{V_{in}(s)} = \frac{1\cdot(Cs)}{(1 + R + Ls\cdot Cs)} = \frac{1}{Ls} \quad \text{in the correct form.}
\]

Set the denominator to 0 and you get the characteristic equation:

\[
s^2 + \frac{R}{L}s + \frac{1}{L\cdot C} = 0
\]

At this point you just proceed with the solution like you did before; Solve the characteristic equation to find \( s_1 \) and \( s_2 \). Decide which case you have (over-, under-, or critically damped). Use the two initial conditions, \( i_L(0) \) and \( v_C(0) \) to find the initial condition and the initial slope of your variable of interest, then use those to find the constants \( B \) and \( D \).

Differential equation from the transfer function

You can also use the transfer function to go back and find the differential equation, just replace each \( s \) with \( \frac{d}{dt} \) and go back to functions of \( t \).

\[
\frac{d}{dt} V_{in}(t) = \left(\frac{d^2}{dt^2} i(t) + \frac{R}{L} \frac{d}{dt} i(t) + \frac{1}{L\cdot C} i(t)\right)
\]

Actually this is a pretty useless thing to do.
ECE 2210

Second-Order Transient Examples

Ex. 1

a) Find the transfer function of the circuit shown. Write your equation in the form of one simple polynomial divided by another

\[
H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{Ls}{\frac{1}{R} + Cs} + \frac{1}{Ls}\left(\frac{1}{\frac{1}{R} + Cs}\right)
\]

\[
= \frac{Ls + LsC^2}{1 + Ls\left(\frac{1}{R} + Cs\right)} = \frac{Ls + LsC^2}{1 + Ls\left(\frac{1}{R} + LsC^2\right)} = \frac{Ls + LsC^2}{LsC^2 + \frac{1}{Ls}} = \frac{s^2 + \frac{1}{C-R}}{s^2 + \frac{1}{Ls}}
\]

b) Find the characteristic equation

\[0 = s^2 + \frac{1}{C-R} + \frac{1}{Ls}\]

Ex. 2

a) Find the characteristic equation of the circuit shown (after the switch moves to the lower position at \(t = 0\)).

\[
\frac{V_C(s)}{V_{in}(s)} = H(s) = \frac{\frac{1}{Ls + C^2}}{\frac{1}{Ls} + R_2\left(\frac{1}{Ls} + C^2\right)} = \frac{1}{\frac{1}{Ls} + \frac{R_2}{Ls} + \frac{R_2C}{L^2}}
\]

\[
= \frac{Ls}{Ls + R_2 + \frac{R_2C}{L^2}} = \frac{1}{R_2C} + \frac{1}{C} + \frac{1}{L^2}
\]

b) Find the solutions of the characteristic equation.

\[s_1 = \frac{-\frac{1}{R_2C} + \frac{1}{L^2}}{2} = -5 \times 10^4 \cdot \text{sec}^{-1}\]

\[s_2 = \frac{-\frac{1}{R_2C} + \frac{1}{L^2}}{2} = -5 \times 10^4 \cdot \text{sec}^{-1}\]

\[s_1 = s_2 \text{ so... critically damped}\]

b) Find the characteristic equation

\[0 = s^2 + \frac{1}{R_2C} + \frac{1}{L^2}\]

Second-Order Transient Examples, p.1
Final conditions:

\[ v_C(\infty) = 0 \text{V} \quad i_L(\infty) = \frac{6 \text{V}}{125 \Omega} = 48 \text{mA} \]

\( v_C(t) = v_C(0) + B \cdot e^{s_1 t} + D \cdot e^{s_2 t} \)

\[ s_1 = -6.25 \times 10^3 + 4.961 \times 10^4 j \quad \omega = \text{Im}(s_1) = 4.961 \times 10^4 \text{sec}^{-1} \]

\[ s_2 = -6.25 \times 10^3 - 4.961 \times 10^4 j \quad \omega = \text{Re}(s_1) = -6.25 \times 10^3 \text{sec}^{-1} \]

Ex. 2 with bigger \( R_2 \)

a) Find the characteristic equation of the circuit shown (after the switch moves to the lower position at \( t = 0 \)).

As before:

\[ \frac{V_C(s)}{V_{in}(s)} = H(s) = \frac{1}{R_2 C s} + \frac{1}{R_2 C s + \frac{1}{L} + s^2} \]

Characteristic equation:

\[ 0 = s^2 + \frac{1}{R_2 C} s + \frac{1}{L C} \]

\[ \frac{1}{R_2 C} = 1.25 \times 10^4 \text{sec}^{-1} \quad \frac{4}{L C} = 1 \times 10^{10} \text{sec}^{-2} \]

b) Find the solutions of the characteristic equation.

\[ s_1 = -6.25 \times 10^3 + 4.961 \times 10^4 j \quad s_2 = -6.25 \times 10^3 - 4.961 \times 10^4 j \]

\[ \alpha = \text{Re}(s_1) = -6.25 \times 10^3 \text{sec}^{-1} \quad \omega = \text{Im}(s_1) = 4.961 \times 10^4 \text{sec}^{-1} \]
c) Find initial and final conditions for \( v_C(t) \)

See drawings above
\[
\begin{align*}
  v_C(0) &= 0 \text{ V} \\
  i_L(0) &= \frac{18 \text{ V}}{25 \Omega + 1 \text{k}\Omega} = 17.561 \text{ mA} \\
  i_C(0) &= \frac{6 \text{ V}}{1 \text{k}\Omega} - 17.561 \text{ mA} = -11.561 \text{ mA} \\
  v_C(\infty) &= 0 \text{ V}
\end{align*}
\]

\[
\frac{\text{d}}{\text{d}t} v_C(0) = \frac{-11.561 \text{ mA}}{C} = -1.445 \times 10^5 \frac{\text{V}}{\text{sec}}
\]

d) Find the full expression of \( v_C(t) \).

**Underdamped**
\[
v_C(t) = v_C(\infty) + e^{\alpha t} (B \cos(\omega t) + D \sin(\omega t))
\]

\[
\begin{align*}
  B &= v_C(0) - v_C(\infty) = 0 \text{ V} - 0 \text{ V} = 0 \text{ V} \\
  D &= \frac{\frac{\text{d}}{\text{d}t} v_C(0) - B \alpha}{\omega} = \frac{-1.445 \times 10^5 \frac{\text{V}}{\text{sec}} - 0 \text{ V} \cdot \alpha}{\omega} = -2.913 \cdot \text{V} \\
  v_C(t) &= -2.913 \cdot \text{V} \cdot e^{\frac{625}{17.561 \text{V} / \text{sec}} \cdot \sin\left(\frac{49610}{17.561 \text{sec}} \cdot t\right)}
\end{align*}
\]

Ex. 3  a) Find the characteristic equation of the circuit shown. (after the switch opens at \( t = 0 \)). Write your equation in the form of a simple polynomials.

\[
H(s) = \frac{I(s)}{V_{in}(s)} = \frac{1}{Z(s)} = \frac{1}{1 + \frac{1}{1 + \frac{1}{R_2 + C_s}}} = \frac{1}{1 + L \cdot s + R_1 \left( \frac{1}{R_2 + C_s} \right)}
\]

\[
= \frac{1}{R_2 + C_s} = \frac{1}{1 + L \cdot s + R_1 \left( \frac{1}{R_2 + C_s} \right)} = \frac{1}{1 + \frac{L}{R_2} + \frac{C}{L} \cdot s^2 + \frac{R_1}{R_2} \cdot \frac{1}{1 + \frac{L}{R_2} + \frac{C}{L} \cdot s}}
\]

\[
= \frac{1}{1 + C \cdot R_2 + \frac{1}{L} \cdot s} \cdot \frac{1}{s^2 + \frac{1}{C \cdot R_2} + \frac{1}{L}} = \frac{1}{s^2 + \frac{1}{L} \cdot \frac{1}{R_1} + \frac{1}{L}}
\]

characteristic eq.: \( 0 = s^2 + \frac{1}{C \cdot R_2} + \frac{1}{L} \cdot s^2 + \left( 1 + \frac{R_1}{R_2} \right) \cdot \frac{1}{L \cdot C} \)
b) Find the solutions (numbers) of the characteristic equation:

\[
b := \frac{1}{C \cdot R_2} + \frac{R_1}{L}
\]

\[
b = 3.5 \cdot 10^4 \cdot \text{sec}^{-1}
\]

\[
s_1 := -\frac{b + \sqrt{b^2 - 4 \cdot k}}{2}
\]

\[
s_1 = -1.75 \cdot 10^4 + 3.455 \cdot 10^4 j
\]

\[
s_2 := -\frac{b - \sqrt{b^2 - 4 \cdot k}}{2}
\]

\[
s_2 = -1.75 \cdot 10^4 - 3.455 \cdot 10^4 j
\]

Underdamped

\[
k := \left(1 + \frac{R_1}{R_2}\right)
\]

\[
k \cdot \frac{1}{L \cdot C}
\]

\[
k = 1.5 \cdot 10^9 \cdot \text{sec}^{-2}
\]

\[
a := \frac{b}{2}
\]

\[
a = -1.75 \cdot 10^4 \cdot \text{sec}^{-1}
\]

\[
\omega := \frac{1}{2} \sqrt{4 \cdot k - b^2}
\]

\[
\omega = 3.455 \cdot 10^4 \cdot \text{sec}^{-1}
\]

Underdamped

\[
\times \text{ pole}
\]

\[
\times \text{ pole}
\]

The poles are the s’s where the denominator is zero, that is, the s₁ & s₂ solutions to the characteristic equation.

The zero is the s where the numerator is zero:

\[
0 = \frac{1}{L \cdot C \cdot R_2} + \frac{C}{L \cdot C} \cdot s
\]

\[
s = -\frac{1}{C \cdot R_2} = -1 \cdot 10^4 \cdot \text{sec}^{-1}
\]

c) Plot the poles and zeroes of the transfer function.

d) Initial and final conditions for \(i_L(t)\) and \(v_C(t)\).

Before the switch opens

\[
V_{\text{in}} = 12 \cdot \text{V}
\]

\[
R_1 = 200 \cdot \Omega
\]

\[
i_L(0) = \frac{V_{\text{in}}}{R_1} = 60\cdot \text{mA}
\]

\[
v_C(0) = 0
\]

Final condition:

\[
i_L(\infty) = \frac{V_{\text{in}}}{R_1 + R_2} = 10\cdot \text{mA}
\]

\[
v_C(\infty) = \frac{R_2}{R_1 + R_2} \cdot V_{\text{in}} = \frac{1\cdot \text{k}\Omega}{200 \cdot \Omega + 1\cdot \text{k}\Omega} \cdot 12 \cdot \text{V} = 10 \cdot \text{V}
\]
e) Find the full expression of $i_L(t)$.

**Underdamped**

\[ X(t) = X(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega t) + D \cdot \sin(\omega t)) \]

\[ i_L(t) = i_L(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega t) + D \cdot \sin(\omega t)) \]

\[ i_L(0) = i_L(\infty) + B \quad \text{so..} \quad B = i_L(0) - i_L(\infty) \]

\[ B = 60 \cdot mA - 10 \cdot mA \quad B = 50 \cdot mA \]

\[ \frac{d}{dt} i_L(0) = B \cdot \alpha + D \cdot \omega \quad \text{so..} \quad D = \frac{\frac{d}{dt} i_L(0) - B \cdot \alpha}{\omega} \]

\[ D := \frac{0 - B \cdot \alpha}{\omega} \quad D = 25.325 \cdot mA \]

\[ i_L(t) = 10 \cdot mA + e^{\frac{17500}{t}} \cdot \left( 50 \cdot mA \cdot \cos \left( \frac{34550}{sec} \cdot t \right) + 25.325 \cdot mA \cdot \sin \left( \frac{34550}{sec} \cdot t \right) \right) \]

\[ i_L(\infty) = 10 \cdot mA \]

---

f) Find the full expression of $v_C(t)$.

\[ B := 0 \cdot V - 10 \cdot V \quad B = -10 \cdot V \]

\[ v_C(t) = v_C(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega t) + D \cdot \sin(\omega t)) \]

\[ v_C(t) = \frac{6 \cdot 10^5}{sec} V - B \cdot \alpha \]

\[ D := \frac{12.301}{sec} \quad D = 12.301 \cdot V \]

\[ v_C(0) = 10 \cdot V + e^{\frac{17500}{t}} \cdot \left( -10 \cdot V \cdot \cos \left( \frac{34550}{sec} \cdot t \right) + 12.301 \cdot V \cdot \sin \left( \frac{34550}{sec} \cdot t \right) \right) \]

\[ v_C(\infty) = 10 \cdot V \]
h) What value of $R_1$ would make this system critically damped?

\[
\left(\frac{1}{C \cdot R_2} + \frac{R_1}{L}\right)^2 = 4 \left(1 + \frac{R_1}{R_2}\right) \frac{1}{L \cdot C}
\]

\[
0 = \left(\frac{1}{L^2} R_1^2 - \frac{2}{L \cdot C \cdot R_2} R_1\right) + \frac{1}{C^2 \cdot R_2^2} \frac{4}{L \cdot C}
\]

Solve for $R_1$ with quadratic equation:

\[
R_1 = \frac{2 \cdot L + \sqrt{\frac{2 \cdot L^2}{C \cdot R_2} - 4 \cdot \frac{L^2}{C^2 \cdot R_2^2} - 4 \cdot \frac{L}{C}}}{2}
\]

Quadratic equation can be reduced to:

\[
= \frac{L}{C \cdot R_2} - \frac{4}{2} \sqrt{\frac{L}{C}} = -485.7 \cdot \Omega \quad \text{this solution can't be}
\]

\[
= \frac{L}{C \cdot R_2} + \frac{4}{2} \sqrt{\frac{L}{C}} = 645.7 \cdot \Omega \quad \text{this must be the solution}
\]

**Ex. 2 with bigger $R_1$**

$R_1 := 1 \cdot k\Omega$ This should make the system overdamped

\[
b = \frac{1}{C \cdot R_2} + \frac{R_1}{L} \quad b = 1.35 \cdot 10^5 \cdot \text{sec}^{-1}
\]

\[
k = 1 + \frac{R_1}{R_2} \quad k = 2.5 \cdot 10^5 \cdot \text{sec}^{-2}
\]

\[
s_1 := \frac{-b + \sqrt{b^2 - 4 \cdot k}}{2} \quad s_1 = -2.215 \cdot 10^4 \cdot \text{sec}^{-1}
\]

\[
s_2 := \frac{-b - \sqrt{b^2 - 4 \cdot k}}{2} \quad s_2 = -1.128 \cdot 10^5 \cdot \text{sec}^{-1}
\]

Overdamped

\[
v_C(0) = 0 \quad \frac{d}{dt} v_C(0) = \frac{i(0)}{C} = 12 \cdot \text{mA} = i_C(0)
\]

\[
v_C(\infty) = \frac{V_{\text{in}}}{R_1 + R_2} = 6 \cdot V \quad i_L(\infty) = \frac{V_{\text{in}}}{R_1 + R_2} = 6 \cdot \text{mA}
\]

\[
v_C(0) = v_C(\infty) + B + D
\]

\[
0 \cdot V = 6 \cdot V + B + D \quad B = -(6 \cdot V + D)
\]

\[
\frac{d}{dt} v_C(0) = B \cdot s_1 + D \cdot s_2 = -6 \cdot V \cdot s_1 - D \cdot s_1 + D \cdot s_2 \quad D := \frac{1.2 \cdot 10^5 \frac{V}{\text{sec}} + 6 \cdot V \cdot s_1}{s_2 - s_1}
\]

\[
B = -6.143 \cdot V
\]

\[
v_C(t) = v_C(\infty) + B \cdot e^{s_1 t} + D \cdot e^{s_2 t}
\]

\[
v_C(t) = 6 \cdot V - 6.143 \cdot V \cdot e^{1.215 \cdot 10^4 \frac{V}{\text{sec}}} + 0.143 \cdot V \cdot e^{1.128 \cdot 10^5 \frac{V}{\text{sec}}}
\]

\[
v_C(\infty) = 6 \cdot V
\]
Laplace impedances

Resistor | Capacitor | Inductor
---|---|---
\( \frac{1}{R} \) | \( C \) | \( -\frac{1}{L} \)
\( Z_R = R \) | \( Z_C = \frac{1}{C \cdot s} \) | \( Z_L = L \cdot s \)

Transfer function

Use Laplace impedances, manipulate your circuit equation(s) to find a transfer function:

\[
H(s) = \frac{\text{output}}{\text{input}} = \frac{V_X(s)}{V_{in}(s)} = \frac{a_1 \cdot s^2 + b_1 \cdot s + k_1}{s^2 + b \cdot s + k}
\]

May be \( I_X \) or any desired variable
\( a_1, b_1, k_1 \) coefficients may be zero

Rearrange circuit equation to:

\[
\frac{V_X(s)}{V_{in}(s)} = \frac{a_1 \cdot s^2 + b_1 \cdot s + k_1}{s^2 + b \cdot s + k}
\]

Characteristic equation

To find the poles of the transfer function

\[
s^2 + b \cdot s + k = 0
\]

Complete solution

Solutions to the characteristic equation:

\[
s_1 = -\frac{b}{2} + \frac{\sqrt{b^2 - 4 \cdot k}}{2}, \quad s_2 = -\frac{b}{2} - \frac{\sqrt{b^2 - 4 \cdot k}}{2}
\]

Find initial Conditions (\( v_C \) and/or \( i_L \))

Find conditions of just before time \( t = 0 \). \( v_C(0-) \) and \( i_L(0-) \). These will be the same just after time \( t = 0 \), \( v_C(0+) \) and \( i_L(0+) \) and will be your initial conditions.

Use normal circuit analysis to find your desired variable: \( v_X(0) \) or \( i_X(0) \)

Also find: \( \frac{d}{dt}v_X(0) \) or \( \frac{d}{dt}i_X(0) \) The trick to finding these is to see that:

\[
\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C} \quad \text{and} \quad \frac{d}{dt}i_L(0) = \frac{v_L(0)}{L}
\]

Find final conditions ("steady-state" or "forced" solution)

DC inputs: Inductors are shorts Capacitors are opens Solve by DC analysis \( v_X(\infty) \) or \( i_X(\infty) \)

AC inputs: Solve by AC steady-state analysis using \( j \omega \)

\( X(t) \) may be replaced by \( v_X(t) \), \( i_X(t) \) or any desired variable in the equations below

Overdamped \( b^2 - 4 \cdot k > 0 \)

\[
X(t) = X(\infty) + B \cdot e^{-s_1 \cdot t} + D \cdot e^{-s_2 \cdot t}
\]

\[
X(0) = X(\infty) + B + D \quad \frac{d}{dt}X(0) = B \cdot s_1 + D \cdot s_2
\]

Solve simultaneously for \( B \) and \( D \)

Critically damped \( b^2 - 4 \cdot k = 0 \)

\[
X(t) = X(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot t \cdot e^{s_1 \cdot t}
\]

\[
X(0) = X(\infty) + B
\]

\[
\text{so.. } B = X(0) - X(\infty) \quad \frac{d}{dt}X(0) = B \cdot s + D \quad \text{so.. } D = \frac{d}{dt}X(0) - B \cdot s
\]

Underdamped \( b^2 - 4 \cdot k < 0 \)

\[
X(t) = X(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))
\]

\[
X(0) = X(\infty) + B
\]

\[
\text{so.. } B = X(0) - X(\infty) \quad \frac{d}{dt}X(0) = B \cdot \alpha + D \cdot \omega \quad \text{so.. } D = \frac{d}{dt}X(0) - B \cdot \alpha
\]

ECE 2210 Notes, Second Order Transients
**Ex. 1** For the circuit shown:

a) Find the transfer function \( v_L \).

\[
V_L(s) = \frac{1}{\frac{1}{Ls} + \frac{1}{R} + \frac{1}{C} \cdot s} \cdot V_S(s)
\]

\[
= \frac{1}{\frac{1}{Ls} + \frac{1}{C} \cdot s + \frac{1}{R}} \cdot V_S(s)
\]

\[
H(s) = \frac{V_L(s)}{V_S(s)} = \frac{s^2}{s^2 + \frac{1}{C} \cdot R \cdot s + \frac{1}{L} \cdot \frac{1}{C}}
\]

\[
= \frac{s^2}{s^2 + \frac{3.78 \times 10^{-4}}{\text{sec}} \cdot s + \frac{9.09 \times 10^{-9}}{\text{sec}^2}}
\]

b) Find the characteristic equation for this circuit.

\[
0 = s^2 + \frac{1}{C} \cdot R \cdot s + \frac{1}{L} \cdot \frac{1}{C}
\]

Just the denominator set to zero. The solutions of the characteristic equation are the "poles" of the transfer function.

c) Find the differential equation for \( v_L \).

Cross-multiply the transfer function

\[
s^2 \cdot V_S(s) = \left( s^2 + \frac{1}{C} \cdot R \cdot s + \frac{1}{L} \cdot \frac{1}{C} \right) \cdot V_L(s)
\]

\[
s^2 \cdot V_S(s) = s^2 \cdot V_L(s) + \frac{1}{C} \cdot R \cdot s \cdot V_L(s) + \frac{1}{L} \cdot \frac{1}{C} \cdot V_L(s)
\]

\[
\frac{d^2}{dt^2} v_S(t) = \frac{d^2}{dt^2} v_L(t) + \frac{1}{C} \cdot R \cdot \frac{d}{dt} v_L(t) + \frac{1}{L} \cdot \frac{1}{C} \cdot v_L(t)
\]

\[
\frac{d^2}{dt^2} v_S(t) = \frac{d^2}{dt^2} v_L(t) + \frac{3.78 \times 10^{-4}}{\text{sec}} \cdot \frac{d}{dt} v_L(t) + \frac{9.09 \times 10^{-9}}{\text{sec}^2} \cdot v_L(t)
\]

d) What are the solutions to the characteristic equation?

\[
s_1 = \frac{-3.78 \times 10^{-4}}{2} + \frac{1}{2} \sqrt{\left(3.78 \times 10^{-4}\right)^2 - 4 \cdot \left(9.09 \times 10^{-9}\right)} = -1.894 \times 10^{-4} + 9.345 \times 10^{-4} \cdot j
\]

\[
s_2 = \frac{-3.78 \times 10^{-4}}{2} - \frac{1}{2} \sqrt{\left(3.78 \times 10^{-4}\right)^2 - 4 \cdot \left(9.09 \times 10^{-9}\right)} = -1.894 \times 10^{-4} - 9.345 \times 10^{-4} \cdot j
\]

e) What type of response do you expect from this circuit? The solutions to the characteristic equation are complex so the response will be **underdamped**.
Ex. 2 Analysis of the circuit shown yields the characteristic equation below. The switch has been in the open position for a long time and is closed (as shown) at time $t = 0$. Find the initial and final conditions and write the full expression for $i_L(t)$, including all the constants that you find.

$$s^2 + \left(\frac{1}{C-R_1}\right)s + \left(\frac{1}{L/C}\right) = 0$$

$$\left(\frac{1}{C-R_1}\right) = 1 \cdot 10^4 \cdot \frac{1}{\text{sec}}$$

$$\left(\frac{1}{L/C}\right) = 2 \cdot 10^7 \cdot \frac{1}{\text{sec}^2}$$

$$s^2 + 10000 \frac{1}{\text{sec}} s + 2 \cdot 10^7 \frac{1}{\text{sec}^2} = 0$$

$$s_1 = \left[-\frac{10000}{2} + \frac{1}{2}\sqrt{(10000)^2 - 4 \cdot (2 \cdot 10^7)}\right] \text{sec}^{-1}$$

$$s_1 = -2764 \text{sec}^{-1}$$

$$s_2 = \left[-\frac{10000}{2} - \frac{1}{2}\sqrt{(10000)^2 - 4 \cdot (2 \cdot 10^7)}\right] \text{sec}^{-1}$$

$$s_2 = -7236 \text{sec}^{-1}$$

$s_1$ and $s_2$ are both real and distinct, overdamped

Find the initial conditions:

Before the switch closed, the inductor current was:

$$i_L(0) = 30 \text{mA}$$

Before the switch closed, the capacitor voltage was:

$$v_C(0) = 9 \text{V}$$

When the switch is closed, the inductor is suddenly in parallel with the capacitor, and:

$$v_L(0) = v_C(0)$$

$$\frac{di_L(0)}{dt} = \frac{1}{L} v_L(0) = \frac{1}{9 \text{V}} \cdot 90 \text{A} / \text{sec}$$

$$\text{Initial conditions: } i_L(0) = \frac{15 \text{V}}{R_1 + R_2} = i_L(\infty) + B + D, \text{ so } B = i_L(0) - i_L(\infty) - D = 30 \text{mA} - 75 \text{mA} - D = -45 \text{mA} - D$$

$$\frac{di_L(0)}{dt} = 90 \frac{\text{A}}{\text{sec}} = s_1 B + s_2 D = s_1 (-45 \text{mA} - D) + s_2 D = s_1 (-45 \text{mA}) - s_1 D + s_2 D$$

solve for $D$ & $B$:

$$D := \frac{90 \frac{\text{A}}{\text{sec}} - s_1 (-45 \text{mA})}{-s_1 + s_2}$$

$$D = 7.69 \text{mA}$$

$$B = -45 \text{mA} - D = -52.7 \text{mA}$$

Plug numbers back in:

$$i_L(t) := 75 \text{mA} - 52.7 \text{mA} e^{2764t} + 7.69 \text{mA} e^{7236t}$$

Initial slope:

$$\frac{i_L(0.00001) - i_L(0)}{0.00001 \text{sec}} = 89.989 \frac{\text{A}}{\text{sec}}$$

Ends at 75mA

Starts at 30mA
Analysis of the circuit shown yields the characteristic equation and s values below. The switch has been in the closed position for a long time and is opened (as shown) at time \( t = 0 \). Find the initial and final conditions and write the full expression for \( v_C(t) \), including all the constants.

\[
0 = s^2 + \frac{R_1}{L}s + \frac{1}{L}C
\]

\[s_1 = \left(-250 + 10^4j\right) \cdot \frac{1}{\text{sec}}, \quad s_2 = \left(-250 - 10^4j\right) \cdot \frac{1}{\text{sec}}\]

**Solution:**

\[
\alpha := -250 \cdot \frac{1}{\text{sec}} \quad \omega := 10000 \frac{\text{rad}}{\text{sec}}
\]

**Initial conditions:**

before switch opens

\[
i_L(0) = \frac{V_{in}}{R_1 + R_2} = 100 \cdot \text{mA}
\]

\[
v_C(0) = V_{in} \cdot \frac{R_2}{R_1 + R_2} = 6 \cdot \text{V}
\]

Find final condition:

just after the switch opens

\[
\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C} = \frac{100 \cdot \text{mA}}{C} = 8 \cdot 10^5 \cdot \frac{\text{V}}{\text{sec}}
\]

Find constants:

\[
v_C(0) = v_C(\infty) + B \quad B = v_C(0) - v_C(\infty) \quad B := 6 \cdot \text{V} - 10 \cdot \text{V} \quad B = 4 \cdot \text{V}
\]

\[
\frac{d}{dt}v_C(0) = \alpha \cdot B + D \cdot \omega
\]

\[
D := \frac{8 \cdot 10^5 \cdot \text{V}}{\sec} \cdot \frac{\omega}{\alpha} \quad D = 79.9 \cdot \text{V}
\]

Write the full expression for \( v_C(t) \), including all the constants that you find.

\[
v_C(t) = e^{\alpha t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t)) + v_C(\infty)
\]

\[
v_C(t) := e^{-250t} \cdot \left(10 - 80 \cdot e^{-250t}\right) + 79.9 \cdot \text{V} \cdot \sin \left(10^4 \cdot \text{t}\right) + 10 \cdot \text{V}
\]

\[
\sqrt{D^2 + B^2} = 80 \cdot \text{V}
\]

\[
v_C(1) = 10 + 80 \cdot e^{-250t}
\]

\[
v_C(1) = 10 - 80 \cdot e^{-250t}
\]
Ex. 4  Ex.3 Backwards, switch closes at \( t = 0 \)

Characteristic eq.: \( 0 = s^2 + \left( \frac{1}{C \cdot R_2} + \frac{R_1}{L} \right) \cdot s + \left( 1 + \frac{R_1}{R_2} \right) \cdot \frac{1}{L \cdot C} \)

\( s_1 = -1.257 \cdot 10^3 \cdot \frac{1}{\text{sec}} \quad s_2 = -1.326 \cdot 10^5 \cdot \frac{1}{\text{sec}} \)

Initial conditions, same as Ex.3 final:

\[
\begin{align*}
V_C(0) &= V_{in} = 10 \cdot \text{V} \\
\end{align*}
\]

Find final condition:

\[
\begin{align*}
\text{before switch opens} \\
\frac{i_1(\infty)}{R_1 + R_2} &= \frac{V_{in}}{R_1 + R_2} = 100 \cdot \text{mA} \\
V_C(\infty) &= \frac{R_2}{R_1 + R_2} \cdot V_{in} = 6 \cdot \text{V} \\
\end{align*}
\]

Find constants:

\[
\begin{align*}
v_C(0) &= v_C(\infty) + B + D, \quad \text{so} \quad B = v_C(0) - v_C(\infty) - D = 10 \cdot \text{V} - 6 \cdot \text{V} - D = 4 \cdot \text{V} - D \\
\frac{dv_C(0)}{dt} &= -1.334 \cdot 10^6 \cdot \frac{V}{\text{sec}} = s_1 \cdot B + s_2 \cdot D = s_1 \cdot (4 \cdot \text{V} - D) + s_2 \cdot D = s_1 \cdot (4 \cdot \text{V}) - s_1 \cdot D + s_2 \cdot D \\
D &= \frac{-1.334 \cdot 10^6 \cdot \frac{V}{\text{sec}}}{-s_1 + s_2} = 10.12 \cdot \text{V} \\
B &= 4 \cdot \text{V} - D \\
B &= -6.12 \cdot \text{V} \\
\end{align*}
\]

\[
\begin{align*}
v_C(t) &= 6 \cdot \text{V} - 6.12 \cdot \text{V} \cdot e^{1257 \cdot t} + 10.12 \cdot \text{V} \cdot e^{132600 \cdot t}
\end{align*}
\]
Ex. 5  Analysis of a circuit (not pictured) yields the characteristic equation below.

\[ 0 = s^2 + 400s + 400000 \]

Further analysis yields the following initial and final conditions:

- \( i_L(0) = 120\,\text{mA} \quad v_L(0) = -3\,\text{V} \quad v_C(0) = 7\,\text{V} \quad i_C(0) = -80\,\text{mA} \)
- \( i_L(\infty) = 800\,\text{mA} \quad v_L(\infty) = 0\,\text{V} \quad v_C(\infty) = 12\,\text{V} \quad i_C(\infty) = 0\,\text{mA} \)

Write the full expression for \( i_L(t) \), including all the constants that you find.

Solution:

\[ \frac{400}{2} = 200 \quad \frac{\sqrt{400^2 - 4 \cdot 400000}}{2} = 600j \]

\( s_1 := (-200 + 600j) \cdot \frac{1}{\text{sec}} \) and \( s_2 := (-200 - 600j) \cdot \frac{1}{\text{sec}} \)

\( \alpha := \text{Re}(s_1) \quad \alpha = -200 \cdot \text{sec}^{-1} \quad \omega := \text{Im}(s_1) \quad \omega = 600 \cdot \text{sec}^{-1} \)

Initial slope:

\[ \frac{d}{dt}i_L(0) = \frac{v_L(0)}{L} = -\frac{3\,\text{V}}{L} = -150 \cdot \frac{\text{A}}{\text{sec}} \]

General solution for the underdamped condition:

\[ i_L(t) = i_L(\infty) + e^{\alpha t} \left( B \cdot \cos(\omega t) + D \cdot \sin(\omega t) \right) \]

Find constants:

\[ i_L(0) = i_L(\infty) + B \quad B = i_L(0) - i_L(\infty) \quad B := 120\,\text{mA} - 800\,\text{mA} \]

\[ B := -680\,\text{mA} \]

\[ \frac{d}{dt}i_L(0) = \alpha B + D \cdot \omega \quad D := \frac{-150 \cdot \frac{\text{A}}{\text{sec}} - \alpha B}{\omega} \quad D := -476.667 \cdot \text{mA} \]

Write the full expression for \( i_L(t) \), including all the constants that you find.

\[ i_L(t) := 800\,\text{mA} + e^{200t} \left( -680\,\text{mA} \cdot \cos(600t) - 477\,\text{mA} \cdot \sin(600t) \right) \]
Analysis of a circuit (not pictured) yields the characteristic equation below.

\[ 0 = s^2 + 800s + 160000 \]

Further analysis yields the following initial and final conditions:

\[
\begin{align*}
    i_L(0) &= 30 \text{ mA} & v_L(0) &= -7 \text{ V} & v_C(0) &= 5 \text{ V} & i_C(0) &= 70 \text{ mA} \\
    i_L(\infty) &= 90 \text{ mA} & v_L(\infty) &= 0 \text{ V} & v_C(\infty) &= 12 \text{ V} & i_C(\infty) &= 0 \text{ mA}
\end{align*}
\]

Write the full expression for \( i_L(t) \), including all the constants that you find.

\[ i_L(t) = ? \]

Include \textbf{units} in your answer

\textbf{Solution:}

\[
\begin{align*}
    s_1 &= -400 \text{ sec}^{-1} \\
    s_2 &= -400 \text{ sec}^{-1}
\end{align*}
\]

s_1 and s_2 are the same, \textbf{critically damped}

Initial slope:

\[
\frac{d}{dt} i_L(0) = \frac{v_L(0)}{L} = \frac{-7 \text{ V}}{L} = -20 \text{ A sec}^{-1}
\]

General solution for the critically damped condition:

\[ i_L(t) = i_L(\infty) + B e^{s_1 t} + D t e^{s_2 t} \]

Find constants:

\[
\begin{align*}
    B &= i_L(0) - i_L(\infty) \\
    B &= 30 \text{ mA} - 90 \text{ mA} \\
    B &= -60 \text{ mA} \\
    D &= -20 \frac{A}{\text{sec}} - B \cdot s_1 \\
    D &= -44 \frac{A}{\text{sec}}
\end{align*}
\]

Write the full expression for \( i_L(t) \), including all the constants that you find.

\[
\begin{align*}
    i_L(t) &= 90 \text{ mA} - 60 \text{ mA} e^{-400 \text{ A sec}^{-1}} - 44 \frac{A}{\text{sec}} t e^{-440 \text{ A sec}^{-1}}
\end{align*}
\]
Ex 1. The switch at right has been in the open position for a long time and is closed (as shown) at time $t = 0$.

a) What are the final conditions of $i_L$ and the $v_C$?

\[ i_L(\infty) = \frac{V_S}{R_3} = 600\, \text{mA} \]
\[ v_C(\infty) = V_S = 24\, \text{V} \]

b) Find the initial condition and initial slope of $i_L$ so that you could find all the constants in $i_L(t)$. Don't find $i_L(t)$ or its constants, just the initial conditions.

Before the switch closes:

\[ i_L(0) = \frac{V_S}{R_1 + R_3} = 150\, \text{mA} \]
\[ v_C(0) = V_S \frac{R_3}{R_1 + R_3} = 6\, \text{V} \]

Just after the switch closes:

\[ v_L(0) = 24\, \text{V} - 6\, \text{V} = 18\, \text{V} \]
\[ \frac{di_L(0)}{dt} = \frac{18\, \text{V}}{L} = 36000\, \text{A/sec} \]
\[ i_C(0) = 150\, \text{mA} + \frac{18\, \text{V}}{R_2} - \frac{6\, \text{V}}{R_3} = 225\, \text{mA} \]

C) Find the initial condition and initial slope of $v_C$ so that you could find all the constants in $v_C(t)$. Don't find $v_C(t)$ or its constants, just the initial conditions.

\[ v_C(0) = V_S \frac{R_3}{R_1 + R_3} = 6\, \text{V} \]
\[ \frac{dv_C(0)}{dt} = \frac{225\, \text{mA}}{C} = 150000\, \text{V/sec} \]
Systems

Now that we’ve developed the concept of the transfer function, we can now develop system block diagrams using blocks which contain transfer functions.

Consider a circuit:

\[ H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{R + L_2s}{R + L_1s + L_2s} = \frac{R + L_2s}{R + (L_1 + L_2)s} = \frac{L_2s + R}{(L_1 + L_2)s + R} \]

This could be represented in as a block operator:

\[ V_{in}(s) \rightarrow \frac{L_2s + R}{(L_1 + L_2)s + R} \rightarrow V_o(s) = V_{in}(s) \cdot H(s) \]

Transfer functions can be written for all kinds of devices and systems, not just electric circuits and the input and output do not have to be similar. For instance, the potentiometers used to measure angular position in the lab servo can be represented like this:

\[ \theta_{in}(s) \rightarrow K_p = 0.7 \frac{V}{\text{rad}} = 0.012 \frac{V}{\text{deg}} \rightarrow V_{out}(s) = K_p \theta_{in}(s) \]

In general:

\[ H(s) = \frac{X_{out}(s)}{X_{in}(s)} \rightarrow X_{in}(s) \rightarrow H(s) \rightarrow X_{out}(s) = X_{in}(s) \cdot H(s) \]

\[ X_{in} \text{ and } X_{out} \text{ could be anything from small electrical signals to powerful mechanical motions or forces.} \]

Two blocks with transfer functions \( A(s) \) and \( B(s) \) in a row would look like this:

\[ X_{in}(s) \rightarrow A(s) \rightarrow X_{in}(s) \cdot A(s) \rightarrow B(s) \rightarrow X_{out}(s) = X_{in}(s) \cdot A(s) \cdot B(s) \]

The two blocks could be replaced by a single equivalent block:

\[ X_{in}(s) \rightarrow A(s) \cdot B(s) \rightarrow X_{out}(s) = X_{in}(s) \cdot A(s) \cdot B(s) \]
Summer blocks can be used to add signals:

\[
X_1(s) + X_2(s) = X_1(s) + X_2(s).
\]

OR

\[
X_1(s) + \sum X_2(s) = X_1(s) + \sum X_2(s).
\]

or subtract signals:

\[
X_1(s) - X_2(s) = X_1(s) - X_2(s).
\]

OR

\[
X_1(s) - \sum X_2(s) = X_1(s) - \sum X_2(s).
\]

A feedback loop system is particularly interesting and useful:

\[
X_{in}(s) \rightarrow \sum \rightarrow A(s) \rightarrow X_{out}(s) \rightarrow \sum \rightarrow B(s) \rightarrow X_{in}(s).
\]

The entire loop can be replaced by a single equivalent block:

\[
X_{in}(s) \rightarrow \sum \rightarrow A(s) \rightarrow X_{out}(s) = A \cdot \left( X_{in} + B \cdot X_{out} \right) = A \cdot X_{in} + A \cdot B \cdot X_{out}.
\]

Note that I've begun to drop the (s)

\[
X_{out} - A \cdot B \cdot X_{out} = A \cdot X_{in},
\]

\[
X_{out} \left(1 - A \cdot B\right) = A \cdot X_{in},
\]

\[
\frac{X_{out}}{X_{in}} = \frac{A}{1 - A \cdot B} = H(s).
\]

The equivalent transfer function

\[
A(s) \cdot B(s) \quad \text{is called the "loop gain" or "open loop gain"}
\]
Negative feedback is more common and is used as a control system:

\[ X_{in} - B \cdot X_{out} \]

A(s) \[ \rightarrow \]

\[ X_{out}(s) = A \cdot (X_{in} - B \cdot X_{out}) \]
\[ = A \cdot X_{in} - A \cdot B \cdot X_{out} \]
\[ X_{out} + A \cdot B \cdot X_{out} = A \cdot X_{in} \]
\[ X_{out} \frac{(1 + A \cdot B)}{X_{in}} = \frac{A}{1 + A \cdot B} = H(s) \]

The equivalent transfer function

This is called a "closed loop" system, whereas a system without feedback is called "open loop". The term "open loop" is often used to describe a system that is out of control.

The servo used in our lab can be represented by:

The term "open loop" is often used to describe a system that is out of control.

\[ \theta_{in} \rightarrow K_p \]
\[ + \]
\[ \sum \]
\[ \theta_{out} \]

\[ K_p = 0.7 \cdot \frac{V}{rad} \]

Potentiometer constant

Motor Position Potentiometer

\[ \theta_{out}(s) \]

H(s) = \[ \frac{G \cdot K_T \cdot K_p}{s \left[ J \cdot L_a \cdot s^2 + (J \cdot R_a + B \cdot m \cdot L_a) \cdot s + (B \cdot m \cdot R_a + K_T \cdot K_V) \right] + K_p \cdot G \cdot K_T} \]

See the appendix to lab 9 for the complete analysis.
Diodes Notes

Diodes are basically electrical check valves. They allow current to flow freely in one direction, but not the other. Check valves require a small forward pressure to open the valve. Similarly, a diode requires a small forward voltage (bias) to "turn on". This is called the forward voltage drop. There are many different types of diodes, but the two that you are most likely to see are silicon diodes and light-emitting diodes (LEDs). These two have forward voltage drops of about 0.7V and 2V respectively.

The electrical symbol for a diode looks like an arrow which shows the forward current direction and a small perpendicular line. The two sides of a diode are called the "anode" and the "cathode" (these names come from vacuum tubes). Most small diodes come in cylindrical packages with a band on one end that corresponds to the small perpendicular line, and shows the polarity, see the picture. Normal diodes are rated by the average forward current and the peak reverse voltage that they can handle. Diodes with significant current ratings are known as "rectifier" or "power" diodes. (Rectification is the process of making AC into DC.) Big power diodes come in a variety of packages designed to be attached to heat sinks. Small diodes are known as "signal" diodes because they're designed to handle small signals rather than power.

Diodes are nonlinear parts
So far in this class we've only worked with linear parts. The diode is definitely NOT linear, but it can be modeled as linear in its two regions of operation. If it's forward biased, it can be replaced by battery of 0.7V (2V for LEDs) which opposes the current flow. Otherwise it can be replaced by an open circuit. These are "models" of the actual diode. If you're not sure of the diode's state in a circuit, guess. Then replace it with the appropriate model and analyze the circuit. If you guessed the open, then the voltage across the diode model should come out less than +0.7V (2V for LEDs). If you guessed the battery, then the current through the diode model should come out in the direction of the diode's arrow. If your guess doesn't work out right, then you'll have to try the other option. In a circuit with multiple diodes (say "n" diodes), there will be 2^n possible states, all of which may have to be tried until you find the right one. Try to guess right the first time.

1 Assume the diode is operating in one of the linear regions (make an educated guess).
2 Analyze circuit with a linear model of the diode.
3 Check to see if the diode was really in the assumed region.
4 Repeat if necessary.

Actual diode curve
The characteristics of real diodes are actually more complicated than the constant-voltage-drop model. The forward voltage drop is not quite constant at any current and the diode "leaks" a little current when the voltage is in the reverse direction. If the reverse voltage is large enough, the diode will "breakdown" and let lots of current flow in the reverse direction. A mechanical check valve will show similar characteristics. Breakdown does not harm the diode as long as it isn't overheated.

Zener diodes are special diodes designed to operate in the reverse breakdown region. Since the reverse breakdown voltage across a diode is very constant for a large range of current, it can be used as a voltage reference or regulator. Zener diodes are also used for over-voltage protection. In the forward direction zeners work the same as regular diodes.

Constant-voltage-drop model
This is the most common diode model and is the only one we'll use in this class. It gives quite accurate results in most cases.

Zener diodes are special diodes designed to operate in the reverse breakdown region. Since the reverse breakdown voltage across a diode is very constant for a large range of current, it can be used as a voltage reference or regulator. Zener diodes are also used for over-voltage protection. In the forward direction zeners work the same as regular diodes.
I recommend that you try some of the DC analysis in the Diode Circuit Examples handout before you proceed here.

**Diodes in AC Circuits**

Diodes are often used to manipulate AC waveforms. We’ll start with some triangular waveforms to get the general idea.

Diode doesn’t conduct until \( v_{in} \) reaches 0.7V, so 0.7V is a dividing line between the two models of the diode.

\[
\text{slope} = \frac{0.7 \cdot V}{t_1} = \frac{V_p}{t_p}
\]

\[
t_1 = \frac{0.7 \cdot V}{V_p} \cdot t_p
\]

When the diode conducts, you’re left with a voltage divider

\[
V_{R2\text{peak}} = \left( V_p - 0.7 \cdot V \right) \frac{R_2}{R_1 + R_2}
\]

Sometimes it’s helpful to figure out what the voltage across the diode would be if it never conducted (light dotted line).

\[
t_1 = \frac{0.7 \cdot V}{V_p} \cdot t_p
\]

\[
\frac{V}{R_1} \cdot t_1
\]

\[
t_2 = t_2 - t_1
\]
Rectifier Circuits & Power Supplies

**Half-wave rectification**
What if the input is a sine wave?

\[ v_{RL} \] is now DC, although a bit bumpy. Some things are better if they’re bumpy, but not roads and not DC voltages.

Rectification is the process of making DC from AC. Usually the AC is derived from the AC wall outlet (often through a transformer) and the DC is needed for electronic circuitry modeled by \( R_L \) here.

A “filter” capacitor (usually a big electrolytic) helps smooth out the bumps, although it sure looks like we could a bit bigger one here. The remaining bumpiness is called “ripple”, \( V_r \) is peak-to-peak ripple.

**Full-wave rectification**
The “center tap” in the secondary of this transformer makes it easy to get full-wave rectification.

The center-tap transformer is also good for making \( \pm \) supplies.

**Bridge**
A “bridge” circuit or “bridge rectifier” can give you full-wave rectification without a center-tap transformer, but now you lose another “diode drop.”

Bridge rectifiers are often drawn like this:
Other Useful Diode Circuits

Simple limiter circuits can be made with diodes. A common input protection to protect circuit from excessive input voltages such as static electricity.

The input to the box marked “sensitive circuit” can’t get higher than the positive supply + 0.7V or lower than the negative supply - 0.7V.

Put a fuse in the $V_{in}$ line and the diodes can make it blow, providing what’s known as “crowbar” protection.

Another example of crowbar protection:

- If the input voltage goes above 16 V, the fuse will blow, protecting the circuitry.
- Or, if the input voltage is hooked up backwards the fuse will blow, protecting the circuitry.

AM detector

Battery Isolator

Like you might find in an RV. One alternator is used to charge two batteries. When the alternator is not charging, the batteries, the circuits they are hooked to should be isolated from one another. If not, then one battery might discharge through the second, especially if second is bad. Also, you wouldn’t want the accessories in the RV to drain the starting battery, or your uncle George from South Dakota might never leave your driveway.

Battery Backup Power

Normally the power supply powers the load through D1. However, if it fails, the load will remain powered by the battery through D2. Finally, D3 and R may be added to keep the battery charged when the power supply is working. These sorts of circuits are popular in hospitals.

Diode Logic Circuits

Actually, both of the previous circuits are logic circuits as well.

"Flyback" Diode

Every time the switch opens the inductor current continues to flow through the diode for a moment. If the diode weren’t there, then the current would arc across the switch.
Basic diode circuit analysis

1. Make an educated guess about each diode’s state.
2. Replace each diode with the appropriate model:
3. Redraw and analyze circuit.
4. Make sure that each diode is actually in the state you assumed:

Note: 0.7V is for silicon junction diodes & will be different for other types. (2V for LED)

If any of your guesses don’t work out right, then you’ll have to start over with new guesses. In a circuit with n diodes there will be 2^n possible states, all of which may have to be tried until you find the right one. Try to guess right the first time.

Ex1

Try reverse-biased, non-conducting model

Try forward-biased, conducting model

Ex2

Try forward-biased, conducting model

Try reverse-biased, non-conducting model

Ex3

Try reverse-biased, non-conducting model

Try forward-biased, conducting model

In each of these examples, my first guess was pretty stupid. I did that intentionally to show the process. I expect that you can make better guess and thus save yourself some work.

ECE 2210 Diode Circuit Examples
### Ex4

#### Circuit Diagram:

- \( R_1 := 1\,\text{kΩ} \)
- \( R_2 := 1\,\text{kΩ} \)
- \( V = 5\,\text{V} \)

#### Analysis

**Assume diode conducts:**

\[ V_{D} := 0.7\,\text{V} = V_{R2} \]

\[ V_{R2} := V_{D} \quad I_2 := \frac{V_{R2}}{R_2} \quad I_2 = 0.7\,\text{mA} \]

\[ V_{R1} := 5\,\text{V} - V_{D} \quad V_{R1} = 4.3\,\text{V} \quad I_1 := \frac{V_{R1}}{R_1} \quad I_1 = 4.3\,\text{mA} \]

We assumed conducting (assuming a voltage), so check the current.

\[ I_D := I_1 - I_2 \quad I_D = 3.6\,\text{mA} > 0 \]

So assumption was correct.

### Ex5

#### Circuit Diagram:

- \( R_1 := 1\,\text{kΩ} \)
- \( R_2 := 1\,\text{kΩ} \)
- \( V = 5\,\text{V} \)

#### Analysis

**Assume diode conducts:**

\[ V_{D} := 2\,\text{V} = V_{R2} \]

\[ V_{R2} := V_{D} \quad I_2 := \frac{V_{R2}}{R_2} \quad I_2 = 2\,\text{mA} \]

\[ V_{R1} := 5\,\text{V} - V_{D} \quad V_{R1} = 3\,\text{V} \quad I_1 := \frac{V_{R1}}{R_1} \quad I_1 = 3\,\text{mA} \]

We assumed conducting (assuming a voltage), so check the current.

\[ I_D := I_1 - I_2 \quad I_D = 1\,\text{mA} > 0 \]

So assumption was correct, but the current is probably too small to create noticeable light.

### Ex6

#### Circuit Diagram:

- \( R_1 := 1\,\text{kΩ} \)
- \( R_2 := 100\,\text{Ω} \)
- \( V = 5\,\text{V} \)

#### Analysis

**Assume diode conducts:**

\[ V_{D} := 0.7\,\text{V} = V_{R2} \]

\[ V_{R2} := V_{D} \quad I_2 := \frac{V_{R2}}{R_2} \quad I_2 = 7\,\text{mA} \]

\[ V_{R1} := 5\,\text{V} - V_{D} \quad V_{R1} = 4.3\,\text{V} \quad I_1 := \frac{V_{R1}}{R_1} \quad I_1 = 4.3\,\text{mA} \]

We assumed conducting (assuming a voltage), so check the current.

\[ I_D := I_1 - I_2 \quad I_D = -2.7\,\text{mA} < 0 \]

So assumption was WRONG!

**Assume diode does not conduct:**

\[ I_D := 0\,\text{mA} \]

\[ V_{R2} := I_2 \cdot R_2 \quad V_{R2} = 0.455\,\text{V} < 0.7\,\text{V} \]

We assumed not conducting (assuming a current), so check the voltage.

Actually, this final check isn’t necessary, since first assumption didn’t work, so this one had to.
You can safety say that diode $D_1$ doesn’t conduct without rechecking later because no supply is even trying to make current flow through that diode the right way.

Assume both $D_2$ and $D_3$ conduct.

Analyze $V_{R1} := V_S - V_{D2} - V_{D2}$

$V_{R1} = 3.6\cdot V$

$I_1 := \frac{V_{R1}}{R_1}$

$I_1 = 3.6\cdot \text{mA}$

$I_2 := \frac{V_{D2}}{R_2}$

$I_2 = 2.333\cdot \text{mA}$

$I_3 := \frac{V_{D3}}{R_3}$

$I_3 = 4.667\cdot \text{mA}$

We assumed $D_1$ & $D_2$ conduct (assumed a voltage), so check currents.

$I_{D2} := I_1 - I_2$ $I_{D2} = 1.267\cdot \text{mA}$ $> 0$, so assumption OK

$I_{D3} := I_1 - I_3$ $I_{D3} = -1.067\cdot \text{mA}$ $< 0$, so assumption wrong

Assume $D_2$ conducts and $D_3$ doesn’t.

Analyze $I_2 := \frac{V_{D2}}{R_2}$

$I_2 = 2.333\cdot \text{mA}$

$I_1 := \frac{V_S - V_{D2}}{R_1 + R_3}$

$I_1 = 3.739\cdot \text{mA}$

Assumed $D_2$ conducts, so check $D_2$ current.

$I_{D2} := I_1 - I_2$ $I_{D2} = 1.406\cdot \text{mA}$ $> 0$, so assumption OK

Assumed $D_3$ doesn’t conduct, so check $D_3$ voltage.

$V_{R3} := I_1 \cdot R_3$ $V_{R3} = 0.561\cdot \text{V}$ $< 0.7\text{V}$, so OK

Once you find one case that works, you don’t have to try any others.

### Zener Diodes

Zener diodes are special diodes designed to operate in the reverse breakdown region. Since the reverse breakdown voltage across the diode is very constant for a large range of current, it can be used as a voltage reference or regulator. Diodes are not harmed by operating in this region as long as their power rating isn’t exceeded. In the forward direction zeners work the same as regular diodes.

Now there are three possible regions of operation:

**Same basic diode circuit analysis**

1. Make an educated guess about each diode’s state.
2. Replace each diode with the appropriate model:
3. Redraw and analyze circuit.
4. Make sure that each diode is actually in the state you assumed:
Zener Diode Circuit Examples

**Ex1** Typical shunt regulator circuit:

![Circuit Diagram]

- $V_S := 10 \text{ V}$
- $R_L := 500 \Omega$
- $V_Z := 4.5 \text{ V}$
- $I_1 := 250 \Omega$
- $I_L := 500 \Omega$
- $I_D$ (current through diode)

**ECE 2210 Diode Circuit Examples p4**

- $V_Z := 4.5 \text{ V}$
- $R_L := 500 \Omega$

Assume conducting in breakdown region

- $V_D := V_Z$
- $I_L := \frac{V_Z}{R_L}$
- $I_L = 9 \text{ mA}$
- $I_1 := \frac{V_S - V_Z}{R_1}$
- $I_1 = 22 \text{ mA}$

Assumed a conducting region, so check the current to see if the current flows in the direction shown.

- $I_D := I_1 - I_L$
- $I_D = 13 \text{ mA} > 0$, so assumption OK

**Ex2** What if $R_L$ is smaller? $R_L := 150 \Omega$

Assume conducting in breakdown region

- $I_1 := \frac{V_S - V_Z}{R_1}$
- $I_1 = 22 \text{ mA}$
- $I_D := I_1 - I_L$
- $I_D = 8 \text{ mA} < 0$, so assumption is **WRONG**!

Circuit "falls out of regulation"

Assume not conducting

- $I_L = I_1 := \frac{V_S}{R_1 + R_L}$
- $I_1 = 25 \text{ mA}$

Assumed a non-conducting region, so check the voltage to see if it's in the right range.

- $V_D := \frac{R_L}{R_1 + R_L} V_S$
- $V_D = 3.75 \text{ V} < V_Z = 4.5 \text{ V}$

so this assumption is OK

**Ex3** What if $V_S$ is smaller instead of $R_L$? $V_S := 6 \text{ V}$

Assume conducting in breakdown region

- $I_1 := \frac{V_S - V_Z}{R_1}$
- $I_1 = 6 \text{ mA}$
- $I_D := I_1 - I_L$
- $I_D = -3 \text{ mA} < 0$, so assumption is **WRONG**!

Circuit "falls out of regulation"

Assume not conducting

- $I_L = I_1 := \frac{V_S}{R_1 + R_L}$
- $I_1 = 8 \text{ mA}$

Assumed a non-conducting region, so check the voltage to see if it's in the right range.

- $V_D := \frac{R_L}{R_1 + R_L} V_S$
- $V_D = 4 \text{ V} < V_Z = 4.5 \text{ V}$

so this assumption is OK
Exam-type Diode Circuit Examples

On an exam, I usually tell you what assumptions to make about the diodes, then you can show that you know how to analyze the circuit and test those assumptions. Since everyone starts with the same assumptions, everyone should do the same work.

In the circuit shown, use the constant-voltage-drop model for the silicon diode.

a) Assume that diode $D_1$ does NOT conduct.
Assume that diode $D_2$ does conduct.

Find $V_{R2}$, $V_{R1}$, $I_{R1}$, & $I_{D2}$, based on these assumptions.
Stick with these assumptions even if your answers come out absurd. Hint: think in nodal voltages.

$$V_{R2} = \underline{\ \ \ \ \ } V$$
$$V_{R1} = \underline{\ \ \ \ \ } V$$
$$I_{R1} = \underline{\ \ \ \ \ } A$$
$$I_{D2} = \underline{\ \ \ \ \ } A$$

Solution to a)

$$V_{R2} := V_2 - 0.7 \, V$$
$$V_{R1} := V_1 - V_{R2}$$
$$I_{R1} := \frac{V_{R1}}{R_1}$$
$$I_{R2} := \frac{V_{R2}}{R_2}$$
$$I_{D2} := I_{R2} - I_{R1}$$

$$V_{R2} = 1.3 \, V$$
$$V_{R1} = 0.5 \, V$$
$$I_{R1} = 10 \, mA$$
$$I_{R2} = 5 \, mA$$
$$I_{D2} = -5 \, mA$$

b) Based on your numbers above, does it look like the assumption about $D_1$ was correct? yes no (circle one)
How do you know? (Specifically show a value which is or is not within a correct range.)

yes $V_{D1} = V_{R1} = 0.5 \, V < 0.7 \, V$

$c$) Based on your numbers above, does it look like the assumption about $D_2$ was correct? yes no (circle one)
How do you know?

no $I_{D2} = -5 \, mA < 0$

d) Based on your answers to b) and c), which (if any) of the following was not correctly calculated in part a.

$circle any number of answers$

$$V_{R2} \quad V_{R1} \quad I_{R1} \quad I_{D2}$$

Circle all in this case
Assume that diode $D_1$ is conducting and that diode $D_2$ is not conducting.

a) Find $V_{R1}$, $I_{R1}$, $I_{R3}$, $I_{D1}$, $V_{R2}$ based on these assumptions.

Do not recalculate if you find the assumptions are wrong.

\[
\begin{align*}
V_{R1} &= \_\_\_\_\_\_ \text{V} \\
I_{R1} &= \_\_\_\_\_\_ \text{mA} \\
I_{R3} &= \_\_\_\_\_\_ \text{mA} \\
I_{D1} &= \_\_\_\_\_\_ \text{mA} \\
V_{R2} &= \_\_\_\_\_\_ \text{V}
\end{align*}
\]

Solution:

\[
\begin{align*}
V_{R1} &= 0.7 \text{V} \\
I_{R1} &= \frac{V_{R1}}{R_1} = \frac{0.7}{200} = 3.5 \text{mA} \\
I_{R3} &= \frac{V_{in} - 0.7 \text{V}}{R_2 + R_3} = \frac{3 - 0.7}{400} = 4.6 \text{mA} \\
I_{D1} &= I_{R3} - I_{R1} = 4.6 - 3.5 = 1.1 \text{mA} \\
I_{R2} &= \frac{V_{R2}}{R_2} = \frac{0.46}{100} = 4.6 \text{mA} \\
V_{R2} &= 0.46 \text{V}
\end{align*}
\]

b) Was the assumption about $D_1$ correct? 

\begin{choices}
yes \hspace{1cm} no
\end{choices}

How do you know? (Specifically show a value which is or is not within a correct range.)

\begin{choices}
yes \hspace{1cm} I_{R2} = 4.6 \text{mA} > 0
\end{choices}

c) Was the assumption about $D_2$ correct? 

\begin{choices}
yes \hspace{1cm} no
\end{choices}

How do you know?

\begin{choices}
yes \hspace{1cm} V_{D2} = V_{R2} = 0.46 \text{V} < 0.7 \text{V}
\end{choices}

d) Based on your answers to b) and c), which (if any) of the following was not correctly calculated in part a.

\begin{choices}
V_{R1} \hspace{1cm} I_{R1} \hspace{1cm} I_{R3} \hspace{1cm} I_{D2} \hspace{1cm} V_{R2}
\end{choices}

\begin{choices}
(circles any number of answers)
\end{choices}

Circle none in this case
A voltage waveform (dotted line) is applied to the circuit shown. Accurately draw the output waveform ($v_o$) you expect to see. Label important times and voltage levels.

If diode doesn’t conduct:

Positive half
Diode conducts at: $0.7\cdot V$ input at time: $\frac{0.7\cdot V}{10\cdot V} \cdot 10\cdot ms = 0.7\cdot ms$

Maximum:

$$v_o := (10\cdot V - 0.7\cdot V) \cdot \frac{R_2}{R_1 + R_2}$$

$$v_o = 6.2 \cdot V$$

Negative half
Diode conducts at: $-4\cdot V$ input at time: $\frac{4\cdot V}{10\cdot V} \cdot 10\cdot ms = 16\cdot ms$

Maximum:

$$v_o := -(10\cdot V - 4\cdot V) \cdot \frac{R_2}{R_1 + R_2}$$

$$v_o = -4 \cdot V$$
A voltage waveform (dotted line) is applied to the circuit shown. Accurately draw the output waveform ($v_o$) you expect to see. Label important times and voltage levels.

If diode doesn't conduct:

$$v_o = \frac{R_2}{R_1 + R_2} v_{in}$$

$$\frac{R_2}{R_1 + R_2} \cdot 10\cdot V = 4\cdot V$$

When:

$$v_{in} = \frac{R_1 + R_2}{R_2} \cdot 2\cdot V$$

$$v_{in} = 5\cdot V \quad \text{at: } 5\cdot ms \quad \text{Diode begins to conduct}$$

When diode conducts:

$$v_o = 2\cdot V$$
Silicon atoms
Silicon atoms each have 4 valence electrons (electrons in their outermost shell). That leaves 4 spaces in the outer shell of 8. This makes silicon a very reactive chemical, like carbon, which has the same valence configuration.

Silicon crystals
Each atom covalently bonds with four neighboring atoms to form a tetrahedral crystal, which we'll represent in 2D.

In the pure, "intrinsic" crystal, practically all the electrons are used in bonds and all the spaces are filled, which leaves almost no electrons free to move and thus no way to make current flow.

By the effects of heat, light and/or large electric fields, a few electrons do break free of the bonds and become "free" carriers. That is, they're free to move about crystal and "carry" an electrical current.

Interestingly, the space that was vacated by the electron also acts like a carrier. This pseudo-carrier is called a "hole" and it acts like a positively charged carrier.

Unless there's a lot of heat or light, the intrinsic silicon is still a very bad conductor.
Silicon is considered a semiconductor.

Doping
Some atoms, like boron and aluminum naturally have 3 valence electrons in their outer shells.
If you replace some of the silicon atoms in a crystal with boron there won't be quite enough electrons to fill the crystalline bond structure and unfilled spaces will act just like free holes. This "doped" silicon crystal is now called an p-type semiconductor. The p refers to the "extra" "positive" carriers.

Some atoms, like arsenic and phosphorus naturally have 5 valence electrons in their outer shells.
If you replace some of the silicon atoms in a crystal with arsenic the 5th electron doesn't fit into the crystalline bond structure and is therefore free to roam about and be a carrier. This "doped" silicon crystal is now called an n-type semiconductor. The n refers to the "extra" negative carriers.
Diode Physics (The simple version)

It turns out that the free carriers are the most important things in the semiconductor crystals, so we can simplify the drawings to show only these free carriers.

**PN Junction**
When a p-type semiconductor is created next to an n-type, some of the free electrons from the n side will cross over and fill some of the free holes on the p side. This makes the p side negatively charged and leaves the n side positively charged. When the voltage across the junction reaches about 0.7 V the electrons find it too difficult to move against the charge and the process stops.

A region near the junction is now depleted of carriers and (surprise) is called the depletion region.

**Reverse bias**
This pn junction is now a diode. If you place an external voltage across the diode in the reverse bias direction, the depletion region gets bigger and no current flows.

This reverse bias region can be used as a heat or light sensor since the only current flow should be due to a few carriers produced by these effects.

The reverse biased diode can also be used as a voltage variable capacitor since it is essentially an insulator (the depletion region) sandwiched between two conducting regions.

**Forward bias**
If you place an external voltage across the diode in the forward bias direction, the depletion region shrinks until your external voltage reaches about 0.7 V. After that the diode conducts freely.
Imagine, if you will, a hydraulic device where the flow in a small pipe controls a valve in a larger pipe. The greater the flow in the small pipe the more it opens the valve in the large pipe. Take a look at the figure to the right. As an engineering student you should immediately see that this could be a useful device. One use might be as a flow-controlled on/off valve (switch). Or, depending on the flows and pressures involved, it could be used as an \textit{amplifier}. That is, it could be used to make some hydraulic signal larger and more powerful. (A signal is a flow or pressure which conveys information and an amplifier is a device which increases the power of a signal.)

The electrical equivalent of this flow-controlled valve is a transistor. Specifically the NPN bipolar junction transistor (BJT). (There are other types.) The symbol for a transistor is shown below. Notice that it's a three-terminal device. That's because the control current (into the base) and the controlled current (into the collector) join together to form a single current out of the bottom (the emitter current). The valve drawn below is a more accurate analogy for the electrical transistor.

\begin{center}
\begin{tikzpicture}[scale=0.8]
\node at (0,0) {Base};
\node at (0,-1) {Emitter};
\node at (0,-2) {Collector};
\draw[->] (0,-0.5) -- (0,-1.5);
\draw[->] (0,-1.5) -- (0,-2.5);
\draw[->] (0,-0.5) -- (0,-1);\node at (0,-0.75) {$I_B$};\node at (0,-1.25) {$I_E$};\node at (0,-2.25) {$I_C$};
\end{tikzpicture}
\end{center}

\textbf{NPN transistor}

A transistor has three terminals-- the base, the collector, and the emitter. The current flow from the collector to the emitter (through the transistor) is controlled by the current flow from the base to the emitter. A small base current can control a much larger collector current. Often they are related by a simple factor, called beta ($\beta$). For a given base current, the transistor will allow $\beta$ times as much collector current. The key word here is \textit{allow}. The transistor doesn't make the current flow-- some outside power source does that. It simply regulates the current like the valve above. Big power transistors usually have $\beta$s between 20 and 100. For little signal transistors, $\beta$ is usually between 100 and 400. Darlington transistors (really two transistors in one package) can have $\beta$s in the 1000s.

A transistor can be used as a current controlled switch. When there's no base current, it's off, like an open switch. When there is a base current, it's on. If something outside of the transistor is limiting the collector current to less than $\beta$ times the base current then the transistor will turn on as much as it can, like a closed switch. A transistor that is off is operating in its "cutoff" region. A transistor that is fully on is operating in its "saturation" region. A transistor that is partially on is in active control of its collector current ($\beta$ times the base current) and is operating in its "active" region. (Note the valve analogy has a problem with the "open" and "closed" terms.)

There are many types of transistors. PNP transistors work like the NPN transistors, except that all the currents and voltages are backwards. Field-effect transistors (FETs) are are controlled by voltage instead of current and come in many varieties. In this class we'll only work with NPN transistors.
Silicon diodes are made of two layers of doped silicon, a P layer is the anode and an N layer is the cathode. A P-N junction is a diode.

Bipolar junction transistors (BJTs) consist of three layers of doped silicon. The NPN transistor has a thin layer of P-doped silicon sandwiched between two layers of N-doped silicon. Each P-N junction can act like a diode. In fact, this is a fairly good way to check a transistor with an ohmmeter (set to the diode setting).

The base-emitter junction always acts like a diode, but because the base is very thin, it makes the other junction act like a controlled valve (you probably don’t want to know the details, so call it magic).

**Transistor Symbols**

![Transistor Symbols](image)

Notice the subscripts

\[ V_{BE} = V_B - V_E \]
\[ V_{CE} = V_C - V_E \]

**Modes or regions of operation**

- **Cutoff (off)**: \( v_{BE} < 0.7V \)
  - \( i_B = 0 \)
  - \( i_C = 0 \)
- **Active (partially on)**: \( v_{BE} \geq 0.7V \)
  - \( i_B > 0 \)
  - \( v_{CE} \geq 0.2V \)
  - \( i_C = \beta_i B = \alpha_i E \)
    - limited by something outside the transistor
- **Saturation (fully on)**: \( v_{BE} \geq 0.7V \)
  - \( i_B > 0 \)
  - \( v_{CE} \leq 0.2V \)
  - \( i_C < \beta_i B \)

**The Transistor as a switch**

One of the most common uses of a transistor is as a current-controlled switch. Transistor switches are the basis for all digital circuits, but that’s probably not where you’ll use the transistor. More likely, you’ll want to control a high-current device, like a motor, with a high-current output from a computer or logic circuit. The small integrated circuit won’t be able to supply enough current to run the motor, so you’ll use a transistor to switch the larger current that flows through the motor. The input is hooked to the base of the transistor. (Often through a current limiting resistor, since \( V_B \) will only be 0.7V when the transistor is on.) A small \( I_B \) can switch on the much larger \( I_C \) and \( V_{CE} \) can be as low as 0.2V.

**V_{CC}**: The terminal marked \( V_{CC} \) above is just a circuit terminal hooked to a power supply, drawn in dotted lines here, but usually not shown at all. Power supply wires, like ground wires are often not shown explicitly on schematics. It makes the schematics a little less cluttered and easier to read.

**Diode**: If you’re switching an inductive load, like a motor, you should add a diode so that you’re not trying to switch off the motor current instantly. The diode (called a *flyback* diode when used like this) provides a path for the current still flowing through the motor when the transistor is switched off.
**H-bridge:** Of course, if you want to make the motor turn in both directions you’ll need a more complex circuit. Look at the circuit at right, it’s has the shape of an H, hence the name. If transistors Q₁ and Q₄ are on, then the current flows as shown, left-to-right through the motor. If transistors Q₂ and Q₃ are on, then the current flows the other way through the motor and the motor will turn in the opposite direction. (The motor here is a permanent-magnet DC motor.) In my circuit, the top two transistors are PNP’s, which makes the circuit more efficient. The H-bridge could also be made with all NPN’s or with power MOSFET transistors.

An H-bridge requires four inputs, all operated in concert. To turn on Q₁ and Q₄, as shown, V_{in1} would have to be low and V_{in4} would have to be high. At the same time, the other two transistors would have to be off, so V_{in2} would have to be high and V_{in3} would have to be low.

If the control circuit makes a mistake and turns on Q₁ and Q₃ (or Q₂ and Q₄) at the same time you’ll have a toaster instead of a motor driver, at least for a short while.

The circuit at left requires only two inputs. Transistors Q₅ and Q₆ work as *inverters*, when their inputs are high, their outputs are low and vice-versa. The resistors are known as *pull-up* resistors.

The H-bridge should also include flyback diodes.

**Linear Amplifiers**

The objective of a linear amplifier is to output a faithful reproduction of an input signal, only bigger. A voltage amplifier makes the signal voltage bigger. A current amplifier makes the signal current bigger. Many amplifiers do both. All amplifiers should make the signal power bigger (depends somewhat on the load). Of course that means that they need a source of power, generally DC power from a battery or power supply. The signals are usually AC.

Unlike transistor switches, which operate in cutoff and saturation, linear amplifiers must operate in the active region.

**Important relations:** (active region)

\[ v_{BE} = v_B - v_E = 0.7 \text{V} \quad v_{CE} = v_C - v_E > 0.7 \text{V} (\leq 0.2 \text{V if saturated}) \]

\[ i_C = \beta i_B \quad i_C = \alpha i_E \leq i_E \]

**Bias:**
Outside of the active region the input (base current) doesn’t linearly control the output (collector current). To work as a linear amplifier, a transistor must operate in the active region. That means that the transistor must be turned on part way even when there’s no signal at all. Look back at the valve analogy, if small fluctuations in the horizontal pipe flow (i_B) should produce larger but similar fluctuations in the vertical pipe flow (i_C), then there must always be some flow. If either flow ever stops, the horizontal pipe flow (i_B) is no longer in control.

To work in the active region i_B and i_C must be positive for all values of the AC signals. i_B and i_C must be *biased* to some positive DC value. We use capital letters (I_B and I_C) for these DC bias values and lower case letters (i_b and i_c) for the AC signals that will appear as fluctuations of these DC values.

**Transistor Notes (BJT)** p3
The objective of bias then, is to partially turn on the transistor, to turn it, sort-of, half-way on. Now if I twiddle $i_B$, $i_C$ will show a similar, but bigger, twiddle-- that's the whole idea. The transistor should never go into cutoff for any expected input signal, otherwise you'll get clipping at the output. Clipping is a form of distortion, where the output no longer looks like the input.

Furthermore, the transistor must not saturate. That will also cause clipping at the output.

Because $\beta$ can vary widely from transistor to transistor of the same part number and $V_{BE}$ changes with temperature, achieving a stable bias can be a bit of a problem. Usually an emitter resistor ($R_E$) is needed to stabilize the bias.

**DC Analysis in the active region**

DC analysis applies to both switching and bias, although the circuits we'll look at here will include an $R_E$ and we'll be working in the active region, meaning they are bias circuits. The key to DC analysis with an $R_E$ is usually finding $V_B$.

The circuit at right shows a typical bias arrangement. The equations below are for that circuit, adapt them as necessary to fit your actual circuit.

**If you can neglect $i_B$:**

Often in quick-and-dirty analysis you can neglect the base current, $i_B$. In that case:

$$V_B = V_{CC} \frac{R_{B2}}{R_{B1} + R_{B2}}, \quad V_E = V_B - 0.7 \cdot V, \quad I_E = \frac{V_E}{R_E} \approx I_C, \quad V_C = V_{CC} - I_C \cdot R_C$$

This assumption is OK if: $R_{B1} || R_{B2} << \beta \cdot R_E$

Quick check: $R_{B1} < 10 \cdot R_E$ and/or $R_{B2} < 10 \cdot R_E$ Should result in <10% error if $\beta = 100$

**If you can't neglect $i_B$:**

Then you need to make a Thevenin equivalent of the base bias resistors.

$$V_{BB} = V_{CC} \frac{R_{B2}}{R_{B1} + R_{B2}}, \quad R_{BB} = \frac{1}{\frac{1}{R_{B1}} + \frac{1}{R_{B2}}}$$  \hspace{1cm} \text{(Thevenin Eq.)}$$

From the base's point-of-view, the emitter resistor will look $(\beta + 1)$ times bigger than it really is. This is because $(\beta + 1)$ times as much current flows through $R_E$ than into the base. We can ignore the fact that the current is bigger if we pretend that the resistor is bigger. That leads to the simplified circuit. (Usually we use $\beta$ as the factor rather than $(\beta + 1)$, after all $\beta$ just isn't that well known anyway.)

$$I_B = \frac{V_{BB} - 0.7 \cdot V}{R_{BB} + \beta \cdot R_E}, \quad I_C = \beta \cdot I_B \approx I_E, \quad V_E = I_E \cdot R_E \approx I_C \cdot R_E, \quad V_B = V_E + 0.7 \cdot V$$

$$V_C = V_{CC} - I_C \cdot R_C$$

OR: $V_B = I_B \cdot R_E + 0.7 \cdot V$, $V_E = V_B - 0.7 \cdot V$, $I_E = \frac{V_E}{R_E} \approx I_C$, $V_C = V_{CC} - I_C \cdot R_C$
Examples, DC (Bias) Analysis

1) Given:

- \( V_B := 3 \) V, regardless of current into base
- \( V_{CC} := 20 \) V
- \( R_C := 10 \) k\( \Omega \)
- \( R_E := 2.7 \) k\( \Omega \)

Find \( I_C, V_C, V_{CE}, \) and \( P_Q \):

Solution:

- \( V_E := V_B - 0.7 \) V, \( V_E = 2.3 \) V
- \( I_E := \frac{V_E}{R_E} \), \( I_E = 0.852 \) mA, \( I_C := I_E \)
- \( V_C := V_{CC} - I_C R_C \), \( V_C = 11.48 \) V
- \( V_{CE} := V_C - V_E \), \( V_{CE} = 9.18 \) V, OK, is in active region
- \( P_Q := V_{CE} I_C \), \( P_Q = 7.82 \) mW

2) Given:

- \( V_{CC} := 10 \) V, \( V_C := 7.0 \) V
- \( R_{B1} := 8 \) k\( \Omega \)
- \( R_{B2} := 2 \) k\( \Omega \)
- \( R_E := 220 \) \( \Omega \)

Find \( V_B, I_C, V_{CE}, I_{RB2}, \) and \( P_Q \):

Solution:

- \( V_B := V_{CC} \frac{R_{B2}}{R_{B1} + R_{B2}} \), \( V_B = 2 \) V
- \( V_E := V_B - 0.7 \) V, \( V_E = 1.3 \) V
- \( I_E := \frac{V_E}{R_E} \), \( I_E = 5.91 \) mA, \( I_C := I_E \)
- \( V_{CE} := V_C - V_E \), \( V_{CE} = 5.7 \) V, > 0.2V, OK, is in active region
- \( P_Q := V_{CE} I_C \), \( P_Q = 33.68 \) mW

3) Given:

- \( V_{CC} := 12 \) V
- \( V_E := 2.0 \) V
- \( V_C := 6 \) V
- \( I_{RB2} := 0.1 \) mA
- \( I_C := 4 \) mA

Find \( R_E, R_C, V_B, I_B, R_{B2}, \) and \( R_{B1} \):

Solution:

- \( V_{CE} := V_C - V_E \), \( V_{CE} = 4 \) V, > 0.2V, is in active region
- \( I_E = I_C \), \( I_E := I_C \), \( R_E := \frac{V_E}{I_E} \), \( R_E = 500 \) \( \Omega \)
- \( R_C := \frac{V_{CC} - V_C}{I_C} \), \( R_C = 1.5 \) k\( \Omega \)
- \( V_B := V_E + 0.7 \) V, \( V_B = 2.7 \) V
- \( I_B := \frac{I_C}{\beta} \), \( I_B = 0.027 \) mA
- \( R_{B2} := \frac{V_B}{I_{RB2}} \), \( R_{B2} = 27 \) k\( \Omega \)
- \( R_{B1} := \frac{V_{CC} - V_B}{I_{RB2} + I_B} \), \( R_{B1} = 73.4 \) k\( \Omega \)
AC Analysis of Common emitter (CE) amplifier

With an $R_E$, any AC signal applied to the base will then also appear just as big at the emitter (just lower by 0.7V DC). The AC signal current through $R_E$, will be about the same as through $R_C$, so the AC signal voltage across $R_C$ will be bigger than that across $R_E$ by the ratio of $R_C/R_E$. Recalling that the signal at the emitter is about the same as the signal at the base...

\[
\text{base to collector AC gain } = \frac{v_c}{v_b} = \frac{R_C}{R_E}
\]

If a capacitor is placed in parallel with $R_E$ then the effective AC resistance in the emitter goes way down and the gain goes way up. In that case we need a way to estimate the AC resistance within the base-emitter junction itself.

This is called the small-signal emitter resistance: 
\[
r_e = \frac{25\text{ mV}}{I_C}
\]

To find the gains when the input has a source resistance and the output is connected to a load resistor, the calculations become a little more complex. **YOU DON'T NEED TO KNOW THE FOLLOWING MATERIAL.**

- $R_E$ is the DC resistance from emitter to ground
- $r_e$ is the AC signal resistance from emitter to ground, may be zero

Input impedance: 
\[
R_i = R_{B1} || R_{B2} || \beta (r_e + R_e)
\]

Output impedance: 
\[
R_o = R_C || r_o \text{ often neglected}
\]

AC collector resistance: 
\[
r_c = R_C || R_L || r_o
\]

$r_c$ is a characteristic of the transistor, and is often neglected

Voltage gain: 
\[
A_v = \frac{V_o}{V_b} = \frac{r_c}{r_e + R_e}
\]

OR: 
\[
\frac{V_o}{V_s} = \frac{R_i}{R_s + R_i} = \frac{r_c}{r_e + R_e + R_L}
\]

Current gain: 
\[
A_i = \frac{i_o}{i_i} = \frac{r_c}{r_e + R_e + R_L} = A_v \frac{R_i}{R_L}
\]

There are several other types of transistor amplifiers, but we won’t look at them here.

**AC Signal Example**

If the $v_s$ signal were applied at the base, an AC signal would also appear at the collector. How much larger would it be? (Voltage gain).

\[
\text{base to collector AC gain } = \frac{v_c}{v_b} = \frac{R_C}{R_E} = 8.33 \text{ times bigger}
\]
Ex. 1

\[ V_{CC} = 12 \text{ V} \]
\[ R_C = 100 \Omega \]
\[ V_B = 0.7 \text{ V} \]
\[ V_{BB} = 5 \text{ V} \]
\[ I_B = \text{?} \]
\[ V_C = \text{?} \]
\[ I_C = \text{?} \]
\[ P_Q = \text{?} \]

The little open circles are connections, in this case to unseen power supplies.

If: \( \beta = 100 \)
\[ I_B = \frac{V_{BB} - 0.7 \text{ V}}{R_B} \]
\[ I_B = 0.86 \text{ mA} \]
\[ I_C = \beta I_B \]
\[ I_C = 86 \text{ mA} \]
\[ V_C = V_{CC} - R_C I_C \]
\[ V_C = 3.4 \text{ V} \]
\[ P_Q = V_C \cdot I_C \]
\[ P_Q = 292.4 \text{ mW} \]

If: \( \beta = 200 \)
\[ I_B = \frac{V_{BB} - 0.7 \text{ V}}{R_B} \]
\[ I_B = 0.86 \text{ mA} \]
\[ I_C = \beta I_B \]
\[ I_C = 172 \text{ mA} \]
\[ V_C = V_{CC} - R_C I_C \]
\[ V_C = 0.2 \text{ V} \]
\[ P_Q = V_C \cdot I_C \]
\[ P_Q = 118 \text{ mA} \]

Since saturation can depend on \( \beta \), you usually assume a small \( \beta \) when designing a circuit that should saturate (a switching circuit).

If: \( R_C = 50 \Omega \)
\[ I_C = \beta I_B \]
\[ I_C = 172 \text{ mA} \]
\[ V_C = V_{CC} - R_C I_C \]
\[ V_C = 3.4 \text{ V} \]
\[ P_Q = V_C \cdot I_C \]
\[ P_Q = 584.8 \text{ mW} \]

Saturation also depends on \( R_C \) and \( V_{CC} \).

What is the largest value that \( R_B \) could be and still keep the transistor in saturation?
\[ I_{CSat} = \frac{V_{CC} - 0.2 \text{ V}}{R_C} \]
\[ I_{CSat} = 236 \text{ mA} \]
\[ I_B = \frac{I_{CSat}}{\beta} \]
\[ I_B = 1.18 \text{ mA} \]
\[ R_{Bmax} = \frac{5 \text{ V} - 0.7 \text{ V}}{I_B} = 3.644 \text{ k}\Omega \]

Ex. 2

\[ V_B = 6 \text{ V} \]
\[ V_{BB} = 6 \text{ V} \]
\[ I_B = \text{?} \]
\[ I_E = \text{?} \]
\[ R_E = 50 \Omega \]
\[ V_{CC} = 10 \text{ V} \]
\[ P_Q = \text{?} \]
\[ V_E = ? \]

If \( \beta \) is big enough.
\[ I_B = \frac{I_E}{\beta + 1} \]
\[ I_B = 0.5 \text{ mA} \leq \frac{I_E}{\beta} = 0.53 \text{ mA} \]
\[ P_Q = (V_{CC} - V_E) \cdot I_E \]
\[ P_Q = 0.498 \text{ W} \]
Ex.3  If the load must be connected to ground, a PNP transistor is often a better choice. Let's assume a small $\beta$ and saturation and find the $R_B$ necessary.

$$\begin{align*}
\beta &= 20 \\
V_C &= V_{CC} - 0.2 \cdot V \\
R_C &= 15 \Omega \\
I_{C_{sat}} &= \frac{V_C}{R_C} \\
I_B &= \frac{I_{C_{sat}}}{\beta} \\
V_B &= V_{CC} - 0.7 \cdot V \\
R_B &= \frac{V_B}{I_B} \\
P_Q &= 0.2 \cdot V \cdot I_C \\
V_C &= 19.8 \cdot V \\
I_{B1c} &= 86 \cdot mA \\
V_{B2c} &= 0.2 \cdot V \\
I_{B2c} &= 0 \quad I_{RCc} = 493.3 \cdot mA
\end{align*}$$

Ex.4  Sometimes one transistor can't provide enough amplification. Sometimes you want to "invert" the input (make high off and low on).

$$\begin{align*}
V_{B1} &= 0 \quad V_B = 0.7 \cdot V \\
\beta_1 &= 80 \\
R_B &= 4.1 \Omega \\
V_{CC} &= 5 \cdot V \\
I_{B1o} &= 28.6 \cdot mA \\
V_B &= 28.6 \cdot mA \\
I_{C_{sat}} &= 715 \cdot mA \\
V_{C2o} &= 0.2 \cdot V \\
Q_2 must be in saturation: \\
I_{C_{sat}} &= \frac{V_{CC} - V_{C2o}}{R_C} \\
I_{C2o} &= 493.3 \cdot mA
\end{align*}$$

When the switch is open, current flows in through the load resistor, $R_C$. When it is closed, no current flows through the load. This is an example of logical "inversion".
Ex. 5 Modified from F07 Final

A transistor is used to control the current flow through an inductive load (in the dotted box, it could be a relay coil or a DC motor).

a) Assume the transistor is in saturation (fully on) and that switch has been closed for a long time. What is the load current?

\[
I_C = \text{?}
\]

\[
I_{C_{\text{sat}}} = \frac{V_{CC} - 0.2\cdot V}{R_L}
\]

\[
I_{C_{\text{sat}}} = 600\cdot \text{mA}
\]

b) \( \beta = 80 \) find the minimum value of \( V_S \), so that the transistor will be in saturation.

\[
I_{B_{\text{min}}} = \frac{I_{C_{\text{sat}}}}{\beta}
\]

\[
I_{B_{\text{min}}} = 7.5\cdot \text{mA}
\]

\[
V_{\text{Smin}} = I_{B_{\text{min}}} \left( R_S + R_1 \right) + 0.7\cdot V
\]

\[
V_{\text{Smin}} = 2.8\cdot V
\]

Use this \( V_S \) for the rest of the problem.

c) Does the diode in this circuit ever conduct a significant current? If yes, when and how much?

When the switch opens. \( I_{D_{\text{max}}} = I_{C_{\text{sat}}} = 600\cdot \text{mA} \) from part a)

d) You got a bad transistor. \( \beta = 60 \) Find the new \( I_C \), and \( V_{CE} \) and \( P_Q \).

\[
I_C = \text{?}
\]

\[
I_C = \beta \cdot I_{B_{\text{min}}}
\]

\[
I_C = 450\cdot \text{mA}
\]

Now operating in active region

\[
V_{CE} = \text{?}
\]

\[
V_{CE} := V_{CC} - R_L I_C
\]

\[
V_{CE} = 1.4\cdot V
\]

\[
P_Q = \text{?}
\]

\[
P_Q := V_{CE} I_C
\]

\[
P_Q = 0.63\cdot \text{W}
\]

\( \beta = 60 \) Use this for the rest of the problem.

c) Find the minimum value of \( R_L \) so that the transistor will be in saturation.

\[
I_B := \frac{V_{\text{Smin}} - 0.7\cdot V}{R_S + R_1}
\]

\[
I_B = 7.5\cdot \text{mA}
\]

\[
I_{C_{\text{max}}} := \beta \cdot I_B
\]

\[
I_{C_{\text{max}}} = 450\cdot \text{mA}
\]

\[
R_{L_{\text{min}}} := \frac{V_{CC} - 0.2\cdot V}{I_{C_{\text{max}}}}
\]

\[
R_{L_{\text{min}}} = 10.7\cdot \Omega
\]

d) \( R_L \) can't be changed, so find the maximum value of \( R_1 \) so that the transistor will be in saturation.

\[
I_{C_{\text{sat}}} = 600\cdot \text{mA} \quad \text{from part a)}
\]

\[
I_{B_{\text{min}}} := \frac{I_{C_{\text{sat}}}}{\beta}
\]

\[
I_{B_{\text{min}}} = 10\cdot \text{mA}
\]

\[
R_{1_{\text{max}}} = \frac{V_{\text{Smin}} - 0.7\cdot V}{I_{B_{\text{min}}}} - R_S = 10\cdot \Omega
\]
A transistor is used to control the current flow through an inductive load (in the dotted box, it could be a relay coil or a DC motor).

a) $\beta = 25$ Assume the transistor is in the active region, find $I_{sw}$, $I_L$, $V_L$, $V_{EC}$ and $P_Q$.

$$I_B := \frac{V_S - 0.7 \cdot V}{R_S + R_1}$$

$$I_B = 18.6 \, \text{mA} = I_{sw}$$

$$I_L := \beta I_B$$

$$I_L = 465 \, \text{mA}$$

$$R_L := 10 \, \Omega$$

$$V_L := I_L \cdot R_L$$

$$V_L = 4.65 \cdot V$$

$$V_{EC} := V_S - V_L$$

$$V_{EC} = 5.35 \cdot V$$

$$P_Q := V_{EC} \cdot I_L$$

$$P_Q = 2.488 \cdot W$$

b) Was the transistor actually operating in the active region? yes no (circle one) yes

How do you know? (Specifically show a value which is or is not within a correct range.)

$$V_{EC} = 5.35 \cdot V > 0.2 \cdot V$$

c) Find the maximum value of $R_1$, so that the transistor will be in saturation.

If saturated: $V_{EC} := 0.2 \cdot V$

$$I_{Csat} := \frac{V_S - 0.2 \cdot V}{R_L}$$

$$I_{Csat} = 0.98 \, \text{A}$$

$$I_{Bmin} := \frac{I_{Csat}}{\beta}$$

$$I_{Bmin} = 39.2 \, \text{mA}$$

$$R_{1max} = \frac{V_S - 0.7 \cdot V}{I_{Bmin}} - R_S = -63 \, \Omega$$

NOT POSSIBLE

d) $R_1 = 200 \cdot \Omega$ and can’t be changed, find the minimum value of $\beta$ so that the transistor will be in saturation.

$$I_{Csat} = 0.98 \cdot A$$

$$\beta_{min} := \frac{I_{Csat}}{I_B}$$

$$\beta_{min} = 52.7$$

e) How much power is dissipated by the transistor if it has the $\beta$ you found in part d)

$$P_Q := 0.2 \cdot V \cdot I_{Csat}$$

$$P_Q = 0.196 \cdot W$$

f) Does the diode in this circuit ever conduct a significant current? If yes, when and how much?

When the switch opens, $I_{Dmax} = I_{Csat} = 0.98 \cdot A$ from part a)

g) The switch is open for a while. What is the load current ($I_L$) now? 0
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Ex.7  From F13 Final

A transistor is used to control the current flow through an inductive load (in the dotted box, it could be a relay coil or a DC motor).

a) In order for current to flow in through the load, the switch should be:  i) closed  or  ii) open  (Circle one)

b) Assume the switch has been in the position you circled above for a long time.  $I_L$ is 1.3A.  Find the power dissipated by transistor $Q_2$ (neglect base current and $V_{BE}$).

\[
P_{Q2} = \frac{V_{CE2}}{R_L} \times I_L \]

\[
V_{CE2} = V_{CC2} - I_L \cdot R_L = 1.1 \cdot \text{V}
\]

\[
P_{Q2} = V_{CE2} \cdot I_L = 1.43 \cdot \text{W}
\]

c) This is an unacceptable power loss, so you would like to determine the minimum $\beta_2$ needed so that $Q_2$ will be in saturation.  Assume $Q_1$ is also in saturation.  You may assume $I_E = I_C$ for both transistors.  $\beta_{2\text{min}}$ = ?

\[
I_L = \frac{V_{CC2} - 0.2 \cdot \text{V}}{R_L} = 1.6 \cdot \text{A} = I_C
\]

\[
V_{E2} = V_{CC2} - 0.2 \cdot \text{V} = 4.8 \cdot \text{V}
\]

\[
V_{B2} = V_{E2} + 0.7 \cdot \text{V} = 5.5 \cdot \text{V}
\]

\[
V_{C1} = V_{B2} + 0.2 \cdot \text{V} = 5.7 \cdot \text{V}
\]

\[
I_{C1} = \frac{V_{CC1} - V_{C1}}{R_2} = \frac{V_{CC1} - V_{C1}}{R_2}
\]

\[
I_{B2} = I_{C1} \cdot \frac{\beta_1 + 1}{\beta_1} = 57.5 \cdot \text{mA}
\]

\[
\beta_{2\text{min}} = \frac{I_L}{I_{B2}} = 27.826
\]

Better answer

\[
I_{B2} = I_{C1} \cdot \left(\frac{\beta_1 + 1}{\beta_1}\right) = 58.075 \cdot \text{mA}
\]

\[
\beta_{2\text{min}} = \frac{I_L}{I_{B2}} = 26.551
\]

You replace $Q_2$ with a new transistor that has a $\beta$ greater than what you just calculated.

d) How much power is dissipated by the new transistor $Q_2$ (neglect base current and $V_{BE}$)?  $P_{Q2}$ = ?

\[
P_{Q2} = 0.2 \cdot \text{V} \times I_L = 320 \cdot \text{mW}
\]

e) What is the maximum value of $R_1$ needed to saturate $Q_1$?  $\beta_1 = 100$

\[
I_{B1\text{min}} = \frac{I_{C1}}{\beta_1} = 0.575 \cdot \text{mA}
\]

\[
V_{B1} = V_{B2} + 0.7 \cdot \text{V} = 6.2 \cdot \text{V}
\]

\[
R_{1\text{max}} = \frac{V_{CC1} - V_{B1}}{I_{B1\text{min}}} = 3.13 \cdot \text{k}\Omega
\]

f) Does the diode in this circuit ever conduct a significant current?  If yes, when and how much?

When the switch closes.  $I_{D\text{max}} = I_L = 1.6 \cdot \text{A}$ (from part c)
A couple of transistors are used to control the current flow through an inductive load. The switch has been closed, as shown, for a long time.

a) You measure the voltage at each collector (referenced to ground) as shown on the drawing. Find the power dissipated by transistor $Q_2$.

\[ V_{C1} := 5\,\text{V} \quad V_{C2} := 2\,\text{V} \]
\[ I_L := \frac{V_{CC} - 2\,\text{V}}{R_L} \quad I_L = 1.5\,\text{A} \]
\[ P_{Q2} := V_{C2}I_L \quad P_{Q2} = 3\,\text{W} \]

b) Find the $\beta$ of transistor $Q_2$.

\[ V_{R2} := 5\,\text{V} - 0.7\,\text{V} \quad V_{R2} = 4.3\,\text{V} \]
\[ I_{R2} := \frac{V_{R2}}{R_2} \quad I_{R2} = 43\,\text{mA} \]
\[ \beta_2 := \frac{I_L}{I_{R2}} \quad \beta_2 = 34.884 \]

c) Find the $\beta$ of transistor $Q_1$.

\[ I_{R1} := \frac{V_{CC} - 0.7\,\text{V}}{R_1} \quad \beta_1 := \frac{I_{R2}}{I_{R1}} \quad \beta_1 = 58.9 \]

d) Find the minimum $\beta$ for transistor $Q_1$ to be in saturation. $\beta_{1\text{min}}$?

If $Q_1$ is saturated: \[ V_{R2} := V_{CC} - 0.2\,\text{V} - 0.7\,\text{V} \quad V_{R2} = 7.1\,\text{V} \]
\[ I_{R2} := \frac{V_{R2}}{R_2} \quad I_{R2} = 71\,\text{mA} \]
\[ \beta_{1\text{min}} := \frac{I_{R2}}{I_{R1}} \quad \beta_{1\text{min}} = 97.3 \]

You replace $Q_1$ with a different transistor so that now: $\beta_1 := 200$ Use this from now on.

e) Find the new load current ($I_L$) assuming transistor $Q_2$ is in the active region.

$Q_1$ is saturated: \[ I_{R2} = 71\,\text{mA} \quad I_L := I_{R2} \beta_2 \quad I_L = 2.477\,\text{A} \]

f) Check the assumption that $Q_2$ is in the active region and recalculate $I_L$ if necessary.

\[ I_{R2} \beta_2 R_L = 9.907\,\text{V} \quad V_{CE2} := V_{CC} - I_{R2} \beta_2 R_L \quad V_{CE2} = -1.907\,\text{V} \quad \text{Not possible} \]

$Q_2$ is saturated: \[ I_L := \frac{V_{CC} - 0.2\,\text{V}}{R_L} \quad I_L = 1.95\,\text{A} \]

g) Does the diode in this circuit ever conduct a significant current? If yes, when and how much?

When the switch opens. $I_{D\text{max}} = 1.95\,\text{A}$ from part f)
Operational Amplifiers

An operational amplifier is basically a complete high-gain voltage amplifier in a small package. Op-amps were originally developed to perform mathematical operations in analog computers, hence the odd name. They are now made using integrated circuit technology, so they come in the typical multi-pin IC packages. With the proper external components, the operational amplifier can perform a wide variety of "operations" on the input voltage. It can multiply the input voltage by nearly any constant factor, positive or negative, it can add the input voltage to other input voltages, and it can integrate or differentiate the input voltage. The respective circuits are called amplifiers, summers, integrators, and differentiators. Op-amps are also used to make active frequency filters, current-to-voltage converters, voltage-to-current converters, current amplifiers, voltage comparators, etc. etc.. These little parts are so versatile, useful, handy, and cheap that they’re kind of like electronic Lego blocks — although somewhat drably colored.

Op-amp characteristics
Operational amplifiers have several very important characteristics that make them so useful:

1. An op-amp has two inputs and it amplifies the voltage difference between those two inputs. These two inputs are known as the noninverting input, labeled (+), and the inverting input, labeled (-), as shown in Fig. 1. The output voltage is a function of the noninverting input voltage minus the inverting input voltage.

   \[
   v_o = G(v_a - v_b) \quad \text{Where } G = \text{voltage gain of the op-amp.}
   \]

2. The op-amp must be connected to external sources of power (not shown on the drawing above). The output voltage \( (v_o) \) cannot be more positive than the positive power source or more negative than the negative power source. The gain \( (G) \) is very high, typically more than 100,000. Together that means that if the output \( (v_o) \) is in the active range (somewhere between its physical limits, often called “rails”), then \( v_a - v_b \approx 0 \), and \( v_a \approx v_b \). This is a very important point. If you don’t see this, look back at the equation above, \( v_o \) is limited, \( G \) is very big, so \( (v_a - v_b) \) must be very small.

3. In fact, \( v_a - v_b \) must be so small that it’s very difficult to make \( v_a \) & \( v_b \) close enough

   **If the output is:**
   - In active range
     - \(-\text{rail} < v_o < +\text{rail}\)

   **The inputs must be:**
   - \( v_a \approx v_b \)

   **If the inputs are:**
   - \( v_a > v_b \)
   - \( v_a < v_b \)

   **The output must be:**
   - \(+\text{rail}\)
   - \(-\text{rail}\)
without using some *negative feedback*. Negative feedback makes the op-amp maintain \( v_a \approx v_b \) for itself. With the proper negative feedback the op-amp keeps \( v_a \approx v_b \) so close that you can assume that \( v_a = v_b \). Without this negative feedback the op-amp output will almost certainly be at one of its limits, either high or low, i.e. NOT in its active, or *linear*, range. Incidentally, circuits without negative feedback are also useful, but then the output is either high or low (digital) and not linearly related to the input. These types of circuits are called *nonlinear* circuits.

4. Op-amps amplify DC as well as AC.

5. The input currents are almost zero. In more technical terms, the op-amp has very high input impedance. As long as you use reasonable resistor values in your circuits (say \( \leq 1 \text{ M}\Omega \)), you can neglect the input currents.

Simple, isn’t it? OK, so it doesn’t sound so simple yet, but the application of these characteristics really isn’t hard. Let’s look at some circuits.

**Linear Circuits**

Linear circuits employ negative feedback to keep \( v_a \approx v_b \). If a circuit has a connection from the output to the inverting (-) input, then it has negative feedback.

**Voltage follower**

The voltage follower shown in Fig. 2 is probably the simplest linear op-amp circuit. Notice the feedback from the output to the inverting (-) input. If we were to hook the circuit input \( (v_a) \) up to some voltage source, say 2 volts DC, what would happen? If the output was lower than 2 V, then the input voltage difference \( (v_a - v_b) \) would be positive and the huge gain of the op-amp would drive the output higher. If the output was higher than 2 V, then \( v_a - v_b \) would be negative and the output would go down. Very quickly the output voltage \( v_o \) would change until \( v_a - v_b \) becomes very small. Or basically, until \( v_o = 2 \text{ V} \). This is the concept of negative feedback! A fraction (in this case all) of the output voltage is "fed back" to the input in order to control the gain of the op-amp. The op-amp works very hard to maintain a very small difference between the voltages on its inputs. This circuit is known as a voltage follower because the output "follows" the input.

Negative feedback is an important concept. It is used in almost all systems, including all natural systems. A very simple example is the heating system in your house. If the air temperature is too low the thermostat detects a difference between its setting and the air temperature and turns on the heater. When the air temperature reaches the set temperature the thermostat turns off the heater—negative feedback. The servo system that you’ve seen in lab is another example of negative feedback. When the motor position sensor senses a different position than the input position sensor the circuit makes the motor turn in such a way that the difference is minimized and the positions line up.
**Noninverting amplifier**

Now suppose we feed back only a fraction of the output voltage rather than all of it. The method used for this is shown in Fig. 3. \( R_1 \) and \( R_f \) constitute a voltage divider. Remember, the current flowing into an op-amp input is virtually nil, so we can neglect its effect on the voltage divider. This is one of the very nice features of an op-amp. In this circuit, as in the voltage follower, the op-amp works very hard to keep \( v_a - v_b \) very small. Only now \( v_b \) is a fraction of \( v_o \) and the op-amp has to make \( v_o \) that much larger.

\[
\begin{align*}
v_i &= v_a = v_b = \frac{R_1}{R_1 + R_f}v_o
\end{align*}
\]

For all practical purposes:

\[
\begin{align*}
v_o &= \frac{R_1 + R_f}{R_1}v_i = (1 + \frac{R_f}{R_1})v_i
\end{align*}
\]

Notice that by adjusting the ratio of \( R_i \) and \( R_1 \), we can make the gain of the op-amp circuit almost anything we want. Isn’t that neat? The circuit in Fig. 3 is called a noninverting amplifier because the output voltage is in phase with the input voltage; that is, it is *not* inverted. When the input voltage increases, the output voltage will also increase and vice-versa. Yes, noninverting is a double negative and kind-of a dumb name.

**Inverting amplifier**

Before going on, observe that I’ve swapped the positions of the two inputs (− & +) on my op-amp symbol. Either way of drawing the op-amp is OK, whatever makes the whole schematic look better. The noninverting input is on the bottom in this case because it’s hooked to ground. Draw the op-amp so that the surrounding circuitry is clear.

The op-amp output is still connected to the inverting (-) input, so again we have negative feedback. If \( v_b > v_a \) then the output will go down, taking \( v_b \) with it, until \( v_b = v_a \). If \( v_b < v_a \) then the output will go up until \( v_b = v_a \). Negative feedback makes the op-amp do its best to equalize its inputs. In this circuit \( v_a = 0 \), which means that the op-amp will try to keep \( v_b = 0 \) as well. The current into the op-amp is zero, so \( i_{in} \) and \( i_f \) must be the same (\( i_f = i_{in} \)). Using these two ideas together:

\[
\begin{align*}
i_{in} &= \frac{v_{in} - 0}{R_{in}} = i_f = 0 - \frac{v_o}{R_f}
\end{align*}
\]

\[
\begin{align*}
v_o &= -\frac{R_f}{R_{in}}v_{in}
\end{align*}
\]

The minus sign means that \( v_o \) will be inverted with respect to \( v_{in} \), hence the name of this amplifier. When \( v_{in} \) is positive, \( v_o \) is negative, and when \( v_{in} \) is negative, \( v_o \) is positive. The gain of the inverting amplifier, like that of the noninverting amplifier, is completely dependent on our choices of \( R_i \) and \( R_{in} \).
Summer
The inverting amplifier can also be used as a summing amplifier; that is, it can be made to add the effects of several input voltages together. Look at the circuit in Fig. 5.

\[ i_f = i_1 + i_2 + i_3 \]
\[ i_f = \frac{v_o}{R_f} = \frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3} \]
\[ v_o = -\frac{R_f}{R_1}v_1 - \frac{R_f}{R_2}v_2 - \frac{R_f}{R_3}v_3 \]

Differentiator
The differentiator looks an awful lot like the inverting amplifier, and is analyzed in a very similar way.

\[ i_C = C\frac{dv_{in}}{dt} = i_f = -\frac{v_o}{R_f} \]
\[ v_o = -CR_f\frac{dv_{in}}{dt} \]

Integrator
Another useful op-amp circuit is the integrator, shown in Fig. 7. For this circuit:

\[ i_{in} = \frac{v_{in}}{R_{in}} = i_C = -C\frac{dv_o}{dt} \]
\[ v_o = -\frac{1}{CR_{in}}\int v_{in}dt \]

Unfortunately, the simple integrator does have one little practical problem. Notice that if the input voltage has any dc component, the output voltage will soon try to run off to infinity. (Actually it will stop when the op-amp reaches one of its output limits, either negative or positive.) A resistor is usually placed in parallel with the capacitor to eliminate this rather annoying effect. The circuit in Fig. 8 has such a resistor. This is a running-average or Miller integrator.
Active Filters
If you replace the resistors in the inverting and noninverting amplifiers with frequency dependant impedances (capacitors and/or inductors), you can make all sorts of frequency dependant circuits, including filters. In fact, the differentiator and integrator circuits can be thought of as filters.

One of the main advantages of active filters is that you don’t need to use inductors. Real inductors are far from ideal, as you’ve no doubt observed in lab. Real capacitors are much closer to ideal capacitors and they’re cheaper than inductors. Entire books are devoted to these active filters and we won’t cover them any further here.

Differential amplifier
This circuit amplifies only the difference between the two inputs. In this circuit there are two resistors labeled $R_{in}$, which means that their values are equal. Same goes for the two $R_{f}$’s.

$$v_a = \frac{R_f}{R_{in}+R_f}v_2 = \frac{R_f}{R_{in}+R_f}(v_1 - v_o) + v_o$$

$$\frac{R_f}{R_{in}+R_f}v_2 = \frac{R_f}{R_{in}+R_f}(v_1 - v_o) + \frac{R_{in}+R_f}{R_{in}+R_f}v_o$$

$$R_f v_2 = R_f v_1 - R_f v_o + R_{in}v_o + R_f v_o$$

$$R_f v_2 = R_f v_1 + R_{in}v_o$$

$$v_o = \frac{R_f}{R_{in}}(v_2 - v_1)$$

Don’t confuse the differential amplifier with the differentiator. The differential amplifier amplifies the difference of two inputs while the differentiator amplifies the slope of an input.

Instrumentation Amplifier
The differential amplifier isn’t really very practical. The current that flows into the top input depends on the voltage applied to the bottom input. This may not seem that bad, but it is. It means that the input characteristics of this circuit are not constant. One way to get around this would be to place a voltage follower on each input, as shown here.

$$v_o = \frac{R_f}{R_{in}}(v_2 - v_1)$$

Figure 9 Buffered differential amplifier
Now this is a perfectly good circuit. If the two $R_1$s are closely matched and the two $R_{\text{in}}$s are also closely matched, then this circuit will amplify differential voltages very well and reject common voltages (a voltage that is common to both inputs should subtract out of the equation. In EE terms, it has a good Common-Mode-Rejection-Ration (CMRR).

But what if you want to change the gain? You’d have to change two resistors at the same time. By adding two more matched resistors and variable resistor we’ll get the instrumentation amplifier shown at right. The equation for this circuit is:

$$v_o = (1 + \frac{2R_2}{R_1}) \frac{R_4}{R_3} (v_2 - v_1)$$

This is an important circuit and you will probably see it again many times. For instance, if you had to amplify the output of a wheatstone bridge of strain gages, this would be the amp for the job.

**Op-amp with extra current amplification**

Most op-amps cannot supply much current to the load. They are often limited to 10 or 20 mA, about enough to light an LED, but not much more. That can be very limiting. The circuit at right shows a quick and dirty way to use two transistors to greatly increase the load current (at a small cost in output voltage swing). Notice that the feedback is taken from the output of the transistors, so they sort-of become part of the op-amp and the op-amp will do a pretty good job of eliminating the “crossover” dead-zone that occurs as one transistor turns off and the other turns on.

This particular circuit is a simple voltage follower. You can adapt this same current amplification to most of the other op-amp circuits that we have discussed. A few words of warning, however. The extra delay in the feedback can result in instabilities. Try it with the parts you intend to use before you depend on this design. Also, if you use a low quality op-amp (with a slow slew rate) you can get significant crossover distortion.
Nonlinear Circuits
In all cases so far, the feedback signal (voltage) has been applied to the inverting (-) input of the op-amp. This means that the feedback is negative. Negative feedback tends to reduce the difference between the $v_a$ and $v_b$ voltages and make linear circuits. Without negative feedback the op-amp cannot minimize the difference between $v_a$ and $v_b$ and the very high sensitivity of the op-amp results in switching, or nonlinear circuits.

Comparator
Now look at Fig. 14. This circuit will not work as a linear circuit. If $v_a > 0$ the output will be as high as the op-amp can make it, usually a volt or two below the positive power supply. If $v_a < 0$ the output will be as low as the op-amp can make it, usually a volt or two above the negative power supply. The output is no longer linearly related to the input—it’s more like a digital signal, high or low depending on how $v_a$ compares to ground (0 V). The comparator is a nonlinear circuit.

All the circuits above are also comparators. In the first circuit, the input is again compared to ground, but this time the output goes low when the input goes high and vice-versa. In the remaining circuits the input is compared not to 0 V, but to some voltage set by the voltage divider of $R_1$ and $R_2$.

Schmitt trigger
The Schmitt trigger is a variation of the simple comparator which has hysteresis, that is, it has a toggle action. When the output is high, positive feedback makes the switching level higher than it is when the output is low. A little positive feedback makes a comparator with better noise immunity. Increase the positive feedback and the Schmitt trigger can be used in other switching applications.

Look at the Schmitt trigger circuit shown at right. Notice that $v_a = [R_1/(R_1 + R_i)]v_o$, it depends on the output. Lets say...
the output is low and the input is decreasing. When $v_{in} < v_a$ the output goes high and suddenly $v_a$ goes a little bit higher with it. That makes the difference between $v_b$ and $v_a$ even bigger. To make the circuit switch again $v_{in}$ has to go back up beyond the original switching level. It has to reach the new $v_a$ before the output will switch low. In this circuit the two switching levels are above and below ground by the same amount (unless you have nonsymmetric power supplies).

Figure 16 Other Schmitt triggers

The circuits above are variations of the Schmitt trigger. In the first circuit, the input is again compared to levels above and below ground, but this time the output goes high when the input goes high and vise-versa. In the remaining circuits the switching levels are not symmetric about 0 V, but about some voltage set by the voltage divider of $R_1$ and $R_2$.

**Multivibrator** (square wave generator)
The heart of the multivibrator is a Schmitt trigger with lots of positive feedback. Usually $R_2 = R_3$, which set the switching levels at about $\frac{1}{2} V+$ and $\frac{1}{2} V-$. When the output is high the capacitor charges through $R_1$ until it reaches the $\frac{1}{2} V+$ switching level, the output switches low and the capacitor discharges to zero and then charges up (down) until it reaches the $\frac{1}{2} V-$ switching level. That makes the output switch high and the process repeats.

Figure 17 Multivibrator

**Conclusion**
In all of these circuits, with either negative or positive feedback, the output voltage $v_o$ cannot increase without bounds. It is bounded in the positive direction by $V+$, the op-amp positive power supply voltage, and is bounded in the negative direction by $V-$, the op-amp negative supply voltage. If the output voltage is within these bounds, $v_a - v_b$ must be very small. If $v_a - v_b$ were not very small, $v_o$ would soon be forced to one of its limits. Linear circuits use negative feedback to keep this difference small. Without negative feedback you can reasonably assume that the circuit is some kind of switching circuit and that the output is always at one or the other of its limits.

This only scratches the surface of what you can do with op-amps. Get a copy of *The Op-amp Cookbook* for lots more ideas presented in a no-nonsense way.
Could we define an "effective" voltage that would allow us to use the same relationships for AC power as used for DC power?

$$P_{\text{ave}} = \frac{\left(\frac{V_p}{R}\right)^2}{2} = \frac{\left(\frac{V_p}{\sqrt{2}}\right)^2}{2} = \left(\frac{V_p}{\sqrt{2}}\right)^2$$

$$V_{\text{eff}} = \sqrt{\frac{\left(\frac{V_p}{\sqrt{2}}\right)^2}{2}} = \frac{V_p}{\sqrt{2}} = V_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T (v(t))^2 \, dt}$$

RMS Root of the Mean of the Square

Use RMS in power calculations

Sinusoids

$$V_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T (v(t))^2 \, dt} = \sqrt{\frac{1}{T} \int_0^T \left(V_p \cos(\omega t)\right)^2 \, dt} = \sqrt{\frac{1}{T} \int_0^T V_p^2 \left(\frac{1}{2} + \frac{1}{2} \cos(2 \omega t)\right) \, dt}$$

$$= \frac{V_p}{\sqrt{2}} \sqrt{\frac{1}{T} \int_0^T (1) \, dt + \frac{1}{T} \int_0^T \cos(2 \omega t) \, dt} = \frac{V_p}{\sqrt{2}} \sqrt{1 + 0} = \frac{V_p}{\sqrt{2}}$$
Common household power

\[ f = 60 \text{ Hz} \]
\[ \omega = 377 \text{ rad/sec} \]
\[ T = 16.67 \text{ ms} \]

Neutral, N
white
(also ground)

Line, L
black, 120V

Ground, G, green

What about other wave shapes??

Triangular

Square

Works for all types of triangular and sawtooth waveforms

How about AC + DC?

\[
V_{rms} = \sqrt{\frac{1}{T} \int_0^T (v(t))^2 \, dt}
\]

\[
= \sqrt{\frac{1}{T} \int_0^T \left( V_p \cos(\omega t) + V_{DC} \right)^2 \, dt}
\]

\[
= \sqrt{\frac{1}{T} \int_0^T \left[ (V_p \cos(\omega t))^2 + 2 \cdot (V_p \cos(\omega t)) \cdot V_{DC} + V_{DC}^2 \right] \, dt}
\]

\[
= \sqrt{\frac{1}{T} \int_0^T (V_p \cos(\omega t))^2 \, dt + \frac{1}{T} \int_0^T 2 \cdot (V_p \cos(\omega t)) \cdot V_{DC} \, dt + \frac{1}{T} \int_0^T V_{DC}^2 \, dt}
\]

\[
= \sqrt{V_{rmsAC}^2 + V_{DC}^2}
\]

V_{rms} \cdot 120 \, V

V_p = V_{rms} \cdot \sqrt{2} = 170 \, V

Works for all types of triangular and sawtooth waveforms

Same for DC
ECE 2210  AC Power  p3

rectified average

\[ V_{ra} = \frac{1}{T} \int_{0}^{T} |v(t)| \, dt \]

\[ V_{ra} = \frac{2}{\pi} V_p \]

\[ I_{ra} = \frac{2}{\pi} I_p \]

Most AC meters don’t measure true RMS. Instead, they measure \( V_{ra} \), display \( 1.11 V_{ra} \), and call it RMS. That works for sine waves but not for any other waveform.

Some waveforms don’t fall into these forms, then you have to perform the math from scratch

For waveform shown

The average DC (\( V_{DC} \)) value

\[ \frac{2 \cdot V \cdot (4 \cdot ms) + (-1.5 \cdot V) \cdot (2 \cdot ms)}{6 \cdot ms} = -0.333 \cdot V \]

The RMS (effective) value

Graphical way

\[ \frac{4 \cdot V^2 \cdot (4 \cdot ms) + 25 \cdot V^2 \cdot (2 \cdot ms)}{6 \cdot ms} = 11 \cdot V^2 \]

\[ V_{RMS} = \sqrt{11 \cdot V^2} \]

\[ V_{RMS} = 3.32 \cdot V \]

OR...

\[ V_{RMS} = \sqrt{\frac{1}{T} \int_{0}^{T} (v(t))^2 \, dt} \]

\[ = \sqrt{\frac{1}{6 \cdot ms} \left[ \int_{0}^{4 \cdot ms} (2 \cdot V)^2 \, dt + \int_{4 \cdot ms}^{6 \cdot ms} (-5 \cdot V)^2 \, dt \right]} = \sqrt{\frac{1}{6 \cdot ms} \left[ 4 \cdot ms \cdot (2 \cdot V)^2 + 2 \cdot ms \cdot (-5 \cdot V)^2 \right]} = 3.32 \cdot V \]

The voltage is hooked to a resistor, as shown, for 6 seconds.

The energy is transferred to the resistor during that 6 seconds:

\[ P_L = \frac{V_{RMS}^2}{R_L} \]

\[ P_L = 0.22 \cdot W \]

\[ W_L = P_L \cdot 6 \cdot sec \]

\[ W_L = 1.32 \cdot Joule \]

All converted to heat
Use RMS in power calculations

\[ P = I_{\text{Rms}}^2 R = \frac{V_{\text{Rms}}^2}{R} \]

for Resistors ONLY!!

**Capacitors and Inductors**

**Capacitors**

- \[ V_C(t) \]
- \[ i_C(t) \]

**Inductors**

- \[ V_L(t) \]
- \[ i_L(t) \]

Average power is ZERO \( P = 0 \)

Capacitors and Inductors DO NOT dissipate (real) average power.

Reactive power is negative

\[ Q_C = -I_{\text{Rms}} V_{\text{Rms}} \]

\[ = -I_{\text{Rms}} \frac{1}{\omega C} = -V_{\text{Rms}}^2 \omega C \]

Reactive power is positive

\[ Q_L = I_{\text{Rms}} V_{\text{Rms}} \]

\[ = I_{\text{Rms}}^2 \omega L = \frac{V_{\text{Rms}}^2}{\omega L} \]

If current and voltage are not in phase, only the in-phase part of the current matters for the power--- DOT PRODUCT

"Leading" Power

Capacitor dominates

"Lagging" Power

Inductor dominates
Real Power

\[ P = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R} \] for resistors

otherwise...

\[ P = V_{\text{rms}} I_{\text{rms}} \cos(\theta) = I_{\text{rms}}^2 |Z| \cos(\theta) = \frac{V_{\text{rms}}^2}{|Z|} \cos(\theta) \]

units: watts, kW, MW, etc.

\[ P = "\text{Real" Power (average) = } V_{\text{rms}} I_{\text{rms}} \text{pf} = I_{\text{rms}}^2 |Z| \text{pf} = \frac{V_{\text{rms}}^2}{|Z|} \text{pf} \]

Reactive Power

\[ Q = \text{Reactive "power" = } V_{\text{rms}} I_{\text{rms}} \sin(\theta) \]

 capacitors -> - Q  \[ Q_C = I_{\text{rms}}^2 X_C = \frac{V_{\text{rms}}^2}{X_C} \]

\[ X_C = - \frac{1}{\omega C} \] and is a negative number

 inductors -> + Q  \[ Q_L = I_{\text{rms}}^2 X_L = \frac{V_{\text{rms}}^2}{X_L} \]

\[ X_L = \omega L \] and is a positive number

otherwise....

\[ Q = \text{Reactive "power" = } V_{\text{rms}} I_{\text{rms}} \sin(\theta) \]

units: VAR, kVAR, etc. "volt-amp-reactive"

Complex and Apparent Power

\[ S = \text{Complex "power" = } V_{\text{rms}} I_{\text{rms}} \text{ conjugate} = P + jQ = V_{\text{rms}} I_{\text{rms}} \text{ / } \theta \]

units: VA, kVA, etc. "volt-amp"

\[ S = \text{Apparent "power" = } |S| = V_{\text{rms}} I_{\text{rms}} = \sqrt{P^2 + Q^2} \]

units: VA, kVA, etc. "volt-amp"

Power factor

\[ \text{pf} = \cos(\theta) = \text{power factor (sometimes expressed in %)} \]

\[ 0 \leq \text{pf} \leq 1 \]

\[ \theta \] is the **phase angle** between the voltage and the current or the phase angle of the impedance. \[ \theta = \theta_Z \]

\[ \theta < 0 \] Load is "Capacitive", power factor is "leading". This condition is very rare

\[ \theta > 0 \] Load is "Inductive", power factor is "lagging". This condition is so common you can assume any power factor given is lagging unless specified otherwise. Transformers and motors make most loads inductive.

Industrial users are charged for the reactive power that they use, so power factor < 1 is a bad thing.

Power factor < 1 is also bad for the power company. To deliver the same power to the load, they have more line current (and thus more line losses).

Power factors are "corrected" by adding capacitors (or capacitive loads) in parallel with the inductive loads which cause the problems. (In the rare case that the load is capacitive, the pf would be corrected by an inductor.)
Transformer basics and ratings

A Transformer is two coils of wire that are magnetically coupled.

Transformers are only useful for AC, which is one of the big reasons electrical power is generated and distributed as AC.

Transformer turns and turns ratios are rarely given, $V_p/V_s$ is much more common where $V_p$ is the rated primary over rated secondary voltages. You may take this to be the same as $N_1/N_2$ although in reality $N_2$ is usually a little bit bigger to make up for losses. Also common: $V_p : V_s$.

Both RMS

Transformers are rated in VA Transformer Rating (VA) = (rated V) x (rated I), on either side.

Don't allow voltages over the rated V, regardless of the actual current.
Don't allow currents over the rated I, regardless of the actual voltage.

Ideal Transformers

Transformation of voltage and current

$$\frac{N_1}{N_2} = \frac{V_1}{V_2} = \frac{I_2}{I_1}$$

Turns ratio

Turns ratio as defined in Chapman text: $a = \frac{N_1}{N_2}$, same as $N = \frac{N_1}{N_2}$

Note: some other texts define the turns ratio as: $\frac{N_2}{N_1}$

Be careful how you and others use this term

Transformation of impedance

You can replace the entire transformer and load with ($Z_{eq}$). This "impedance transformation" can be very handy.

Transformers can be used for "impedance matching"

This also works the opposite way, to move an impedance from the primary to the secondary, multiply by:
Other Transformers

Multi-tap transformers: Many transformers have more than two connections to primary and/or the secondary. The extra connections are called "taps" and may allow you to select from several different voltages or get more than one voltage at the same time.

Isolation Transformers: Almost all transformers isolate the primary from the secondary. An Isolation transformer has a 1:1 turns ratio and is just for isolation.

Auto Transformers: Auto transformers have only one winding with taps for various voltages. The primary and secondary are simply parts of the same winding. These parts may overlap. Any regular transformer can be wired as an auto transformer. Auto transformers DO NOT provide isolation.

Vari-AC: A special form of auto transformer with an adjustable tap for an adjustable output voltage.

LVDT A Linear-Variable-Differential-Transformers has moveable core which couples the primary winding to the secondary winding(s) in such a way the the secondary voltage is proportional to the position of the core. LVDTs are used as position sensors.

Home power

Standard 120 V outlet connections are shown at right.

The 3 lines coming into your house are NOT 3-phase. They are +120 V, Gnd, -120 V
(The two 120s are 180° out-of-phase, allowing for 240 V connections)

3-Phase Power (FYI ONLY)
Single phase power pulses at 120 Hz. This is not good for motors or generators over 5 hp.

Three phase power is constant as long as the three loads are balanced.

Three lines are needed to transmit 3-phase power. If loads are balanced, ground return current will be zero.

Wye connection:
Connect each load or generator phase between a line and ground.

\[
V_{LN} = \frac{V_{LL}}{\sqrt{3}} \quad I_L = \sqrt{3}I_{LL}
\]

Delta connection:
Connect each load or generator phase between two lines.

\[
V_{LL} = \sqrt{3}V_{LN} \quad I_{LL} = \frac{I_L}{\sqrt{3}}
\]
3-Phase Power (FYI ONLY)

Common 3-phase voltages:

\[ \begin{align*}
208 \, 3\phi & \rightarrow 120-\text{V} \rightarrow 277-\text{V} \\
480 \, 3\phi & \rightarrow 277-\text{V} \rightarrow 120-\text{V}
\end{align*} \]

Apparent Power:

\[ S_{3\phi} = 3 \cdot V_{\text{LN}} \cdot I_{L} = 3 \cdot V_{\text{LL}} \cdot I_{LL} = \sqrt{3} \cdot V_{\text{LL}} \cdot I_{L} \]

Power:

\[ P_{3\phi} = 3 \cdot V_{\text{LN}} \cdot I_{L} \cdot \text{pf} = 3 \cdot V_{\text{LL}} \cdot I_{LL} \cdot \text{pf} = \sqrt{3} \cdot V_{\text{LL}} \cdot I_{L} \cdot \text{pf} = S_{3\phi} \cdot \text{pf} \quad \text{pf} = \cos(\theta) \]

Reactive power:

\[ Q_{3\phi} = 3 \cdot V_{\text{LN}} \cdot I_{L} \cdot \sin(\theta) \quad \text{etc...} = \sqrt{\left( S_{3\phi} \right)^2 - P_{3\phi}^2} \]

\[ I_a = I_L / \alpha \]

\[ V_{an} = V_{\text{LN}} / 0^\circ \]

\[ V_{bn} = V_{\text{LN}} / -120^\circ \]

\[ V_{cn} = V_{\text{LN}} / -240^\circ = V_{\text{LN}} / 120^\circ \]

\[ I_A = I_L / \alpha \]

\[ I_B = I_L / \alpha - 120^\circ \]

\[ I_C = I_L / \alpha - 240^\circ = I_L / \alpha + 120^\circ \]

\[ V_{AB} = V_{\text{LL}} / 30^\circ \]

\[ V_{BA} = V_{\text{LL}} / -90^\circ \]

neutral (ground at some point)

lower-case letters
at source end

upper-case letters
at load end

\[ |V_{AN}| = |V_{BN}| = |V_{CN}| = V_{\text{LN}} = \frac{V_{\text{LL}}}{\sqrt{3}} \]

\[ |I_A| = |I_B| = |I_C| = I_L = \sqrt{3} \cdot I_{LL} \]

\[ |I_{AB}| = |I_{BC}| = |I_{CA}| = I_L = \frac{I_L}{\sqrt{3}} \]

To get equivalent line currents with equivalent voltages

\[ Z_Y = \frac{Z_{\Delta}}{3} \]

\[ Z_{\Delta} = 3 \cdot Z_Y \]
Ex. 1  
R & L together are the load. Find the real power $P$, the reactive power $Q$, the complex power $S$, the apparent power $|S|$, & the power factor $\text{pf}$. Draw phasor diagram for the power.

\[ V_{\text{in}} := 110 \cdot \text{V} \quad R := 10 \cdot \Omega \quad L := 25 \cdot \text{mH} \quad Z := R + j \cdot \omega L \quad \theta := \text{arg}(Z) \quad \theta = 46.7 \cdot \text{deg} \quad \text{pf} := \cos(\theta) \quad \text{pf} = 0.686 \]

\[ Z = 4.704 + 4.991 j \cdot \Omega \quad |Z| = 6.859 \cdot \Omega \quad \theta := \text{arg}(Z) \]

\[ I := \frac{V_{\text{in}}}{Z} \quad I = 11 - 11.671 j \cdot A \quad |I| = 16.038 \cdot A \quad \text{arg}(I) = -46.7 \cdot \text{deg} \]

\[ P := \frac{|V_{\text{in}}|^2}{R} \quad P = 1.21 \cdot \text{kW} \]

\[ Q := \frac{|V_{\text{in}}|^2}{\omega L} \quad Q = 1.284 \cdot \text{kVAR} \]

\[ S := V_{\text{in}} \cdot I \quad \text{OR} \quad S := |S| = \sqrt{P^2 + Q^2} = 1.764 \cdot \text{kVA} \quad \text{pf} = \frac{P}{|S|} = 0.686 \]

What value of $C$ in parallel with $R$ & $L$ would make $\text{pf} = 1$ ($Q = 0$) ?

\[ \text{Im}(I) = -11.671 \cdot A \quad X_C := \frac{V_{\text{in}}}{\text{Im}(I)} \quad X_C = -9.425 \cdot \Omega = \frac{-1}{\omega C} \]

\[ \frac{1}{X_C} \omega = 281 \cdot \mu F \quad \text{OR} \quad \omega = \frac{1}{\sqrt{L \cdot C}} \quad C = \frac{1}{L \cdot \omega^2} \quad C = 281 \cdot \mu F \]

Ex. 2  
R & L together are the load. Find the real power $P$, the reactive power $Q$, the complex power $S$, the apparent power $|S|$, & the power factor $\text{pf}$. Draw phasor diagram for the power.

\[ V_{\text{in}} := 110 \cdot \text{V} \quad R := 10 \cdot \Omega \quad L := 25 \cdot \text{mH} \quad Z := R + j \cdot \omega L \quad \theta := \text{arg}(Z) \quad \theta = 43.304 \cdot \text{deg} \quad \text{pf} := \cos(\theta) \quad \text{pf} = 0.728 \]

\[ I := \frac{V_{\text{in}}}{Z} \quad I = 5.825 - 5.49 j \cdot A \quad |I| = 8.005 \cdot A \quad \text{arg}(I) = -43.304 \cdot \text{deg} \]

\[ P := \frac{|V_{\text{in}}|^2}{R} \quad P = 0.641 \cdot \text{kW} \]

\[ Q := \frac{|V_{\text{in}}|^2}{\omega L} \quad Q = 0.604 \cdot \text{kVAR} \]

\[ S := V_{\text{in}} \cdot I \quad S = 0.641 + 0.604 j \cdot \text{kVA} \quad |S| = 0.881 \cdot \text{kVA} \quad \text{arg}(S) = 43.304 \cdot \text{deg} \quad S = 881 \text{VA/43.3°} \]
OR, if we first find the magnitude of the current which flows through each element of the load...

\[ |I| = \frac{V_{in}}{\sqrt{R^2 + (\omega L)^2}} = 8.005 \ \text{A} \]

\[ P := (|I|)^2 R \quad \quad P = 0.641 \ \text{kW} \quad \quad Q := (|I|)^2 (\omega L) \quad \quad Q = 0.604 \ \text{kVAR} \]

\[ S := P + jQ \quad \quad |S| = \sqrt{P^2 + Q^2} = 0.881 \ \text{kVA} \quad \quad \text{pf} = \frac{P}{|S|} = 0.728 \]

What value of \( C \) in parallel with \( R \) & \( L \) would make \( \text{pf} = 1 \) (Q = 0)?

\[ Q = 603.9 \ \text{kVAR} \quad \quad \text{so we need:} \quad \quad Q_C := -Q \quad \quad Q_C = -603.9 \ \text{kVAR} = \frac{V_{in}^2}{X_C} \]

\[ X_C := \frac{V_{in}^2}{Q_C} \quad \quad X_C = -20.035 \ \Omega \quad \quad C := \frac{1}{|X_C| \omega} \quad \quad C = 132 \ \mu \text{F} \]

Check: \[ \frac{1}{R + j\omega L} = 18.883 \ \Omega \quad \quad \text{No } j \text{ term, so } \theta = 0^\circ \]

**Ex. 3** \( R \), & \( C \) together are the load in the circuit shown.

The RMS voltmeter measures 240 V, the RMS ammeter measures 3 A, and the wattmeter measures 600 W. Find the following: Be sure to show the correct units for each value.

a) The value of the load resistor. \( R_L = ? \)

\[ P = I^2 R_L \]

\[ R_L := \frac{P}{I^2} \quad \quad R_L = 66.7 \ \Omega \]

b) The apparent power. \( |S| = ? \)

\[ S := V_s I \quad \quad S = 720 \ \text{VA} \]

c) The reactive power. \( Q = ? \)

\[ Q := -\sqrt{S^2 - P^2} \quad \quad Q = -398 \ \text{VAR} \]

d) The complex power. \( S = ? \)

\[ S := P + jQ \quad \quad S = 600 - 398i \ \text{VA} \]

e) The power factor. \( \text{pf} = ? \)

\[ \text{pf} := \frac{P}{V_s I} \quad \quad \text{pf} = 0.833 \]

f) The power factor is leading or lagging? \quad \quad \text{leading (load is capacitive, Q is negative)}

g) The two components of the load are in a box which cannot be opened. Add (draw it) another component to the circuit above which can correct the power factor (make \( \text{pf} = 1 \)). Show the correct component in the correct place and find its value. This component should not affect the real power consumption of the load.

Add an inductor in parallel with load

\[ f = 60 \ \text{Hz} \quad \quad \omega := 377 \ \frac{\text{rad}}{\text{sec}} \]

\[ Q = -398 \ \text{VAR} \quad \quad \text{so we need:} \quad \quad Q_L := -Q \quad \quad Q_L = 398 \ \text{VAR} \quad \quad \frac{V_s^2}{X_L} \]

\[ X_L := \frac{V_s^2}{Q_L} \quad \quad X_L = 144.725 \ \Omega \quad \quad L := \frac{|X_L|}{\omega} \quad \quad L = 384 \ \text{mH} \]
**Ex. 4** For the 60 Hz load shown in the figure, the RMS voltmeter measures 120 V. The phasor diagram for the power is also shown. Find the following:

a) The complex power. \( S = ? \)
\[
P = 300 \text{ W} \quad Q = -150 \text{ VA} \quad S = 300 - 150j \text{ VA}
\]

b) The apparent power. \( |S| = ? \)
\[
|S| = \sqrt{P^2 + Q^2} = 335.4 \text{ VA}
\]

c) The power factor. \( \text{pf} = ? \)
\[
\text{pf} = \frac{P}{|S|} = 0.894
\]

d) The item marked "WM" in the figure is a wattmeter, what does it read? (give a number)
\[
P = 300 \text{ W}
\]

e) The item marked "A" in the figure is an RMS ammeter, what does it read? (give a number)
\[
I = 2.795 \text{ A} \quad I = 2.8 \text{ A}
\]

f) The power factor is leading or lagging? leading (Q is negative)

g) The 3 components of the load are in a box which cannot be opened. Add another component to the circuit above which can correct the power factor (make \( \text{pf} = 1 \)). Show the correct component in the correct place and find its value. This component should not affect the real power consumption of the load.

Add an inductor in parallel with load
\[
Q = -150 \text{ VAR} \quad \text{need:} \quad Q_L = -Q \quad Q_L = 150 \text{ VAR} = \frac{V_s^2}{\omega L} \quad L := \frac{V_s^2}{\omega Q_L} \quad L = 255 \text{ mH}
\]

**Ex. 5** R, L, & C together are the load in the circuit shown

The RMS voltmeter measures 120 V. \( V_s := 120 \text{ V} \)
The wattmeter measures 270 W. \( P := 270 \text{ W} \)
The RMS ammeter measures 3.75 A. \( I := 3.75 \text{ A} \)

Find the following: Be sure to show the correct units for each value.

a) The value of the load resistor. \( R_L = ? \)
\[
P = \frac{V_s^2}{R_L} \quad R_L := \frac{V_s^2}{P} \quad R_L = 53.3 \Omega
\]

b) The magnitude of the impedance of the load inductor (reactance). \( |Z_L| = X_L = ? \)
\[
I_R := \frac{V_s}{R_L} \quad I_R = 2.25 \text{ A} \quad I_L := \sqrt{I^2 - I_R^2} \quad I_L = 3 \text{ A} \quad X := \frac{V_s}{I_L} \quad X = 40 \Omega
\]
\[
X_C := -10 \Omega \quad X_L := X - X_C \quad X_L = 50 \Omega
\]

c) The reactive power. \( Q = ? \)
\[
Q := \sqrt{(V_s I)^2 - P^2} \quad Q = 360 \text{ VAR} \quad \text{positive, because the load is primarily inductive}
\]

d) The power factor is leading or lagging? lagging (load is inductive, Q is positive)
e) The 3 components of the load are in a box which cannot be opened. Add another component to the circuit above which can correct the power factor (make pf = 1). Show the correct component in the correct place and find its value. This component should not affect the real power consumption of the load.

Add a capacitor in parallel with load

\[ \text{Q} = 360 \cdot \text{VAR} \quad \text{so we need: } \quad Q_C := -Q \quad Q_C = -360 \cdot \text{VAR} = -\frac{V^2}{s} = -\omega \cdot C \cdot V^2_s \]

\[ C := \frac{Q_C}{-\omega \cdot V^2_s} \]

\[ C = 66.3 \cdot \mu \text{F} \]

**Ex. 6** A step-down transformer has an output voltage of 220 V (rms) when the primary is connected across a 560 V (rms) source.

a) If there are 280 turns on the primary winding, how many turns are required on the secondary?

\[ \frac{280}{560} \cdot \frac{220}{\text{volt}} = 110 \text{ turns} \]

b) If the current in the primary is 2.4 A, what current flows in the load connected to the secondary?

\[ \frac{2.4}{110} \cdot \frac{280}{\text{amp}} = 6.11 \cdot \text{A} \]

c) If the transformer is rated at 700/275 V, 2.1 kVA, what are the rated primary and secondary currents?

\[ \text{pri: } \frac{2.1}{700} = 3 \cdot \text{A} \quad \text{sec: } \frac{2.1}{275} = 7.636 \cdot \text{A} \]

**Ex. 7** The transformer shown in the circuit below is ideal. Find the following:

a) \( |I_1| = ? \)

\[ V_s := 120 \cdot \text{V} \quad \omega := 377 \text{radians} / \text{sec} \]

\[ R_1 := 20 \cdot \Omega \quad N_1 := 150 \text{turns} \]

\[ R_2 := 15 \cdot \Omega \quad N_2 := 50 \text{turns} \]

\[ V_1 \]

\[ \text{C} := 40 \cdot \mu \text{F} \]

\[ Z_L := \frac{1}{R_2} + j \omega \cdot C \]

\[ Z_L = 14.27 - 3.228 j \cdot \Omega \]

\[ Z_{eq} := \left( \frac{N_1}{N_2} \right)^2 \cdot Z_L \quad \text{Z}_{eq} = 128.429 - 29.051 j \cdot \Omega \]

\[ R_1 + Z_{eq} = 148.429 - 29.051 j \cdot \Omega \]

\[ \sqrt{148.429^2 + 29.051^2} = 151.245 \]

\[ |I_1| = \frac{V_s}{R_1 + Z_{eq}} = \frac{V_s}{151.245 \cdot \Omega} = 0.793 \cdot \text{A} \]

b) \( |I_2| = ? \)

\[ \left| \frac{N_1}{N_2} \right| \cdot |I_1| = \frac{150}{50} \cdot 0.793 \cdot \text{A} = 2.379 \cdot \text{A} \]

c) \( |V_1| = ? \)

\[ V_s \left| \frac{Z_{eq}}{R_1 + Z_{eq}} \right| \]

OR:

\[ |V_1| = |I_1| \cdot |Z_{eq}| = 0.793 \cdot \text{A} \cdot \sqrt{128.429^2 + 29.051^2} \cdot \Omega = 104.417 \cdot \text{V} \]