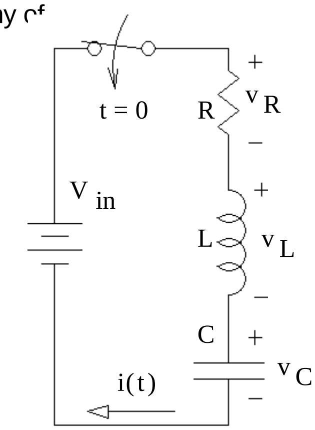
1.6 Second-Order Transients

A circuit with both a capacitor and an inductor is like a mechanical system with both a mass and a spring. When there are two different types of energy-storage elements, the transient responses can be much more interesting than the simple exponential curves that we've seen so far. Many of these systems can oscillate or "ring" when a transient is applied. When you analyze a circuit with a capacitor and an inductor you get a second-order differential equation, so the transient voltages and currents are called second-order transients.



Series RLC circuit, traditional way: Look at the circuit at right. The same current flows through all three elements (i(t) or just i). That current will begin to flow after time t = 0, when the switch is closed. Using basic circuit laws:

$$V_{in} = v_R + v_L + v_C$$

$$= i\cdot R + L\cdot\frac{d}{dt}i + \frac{1}{C}\cdot\int_{-\infty}^{t} i_C\,dt \qquad \text{Making the obvious substitutions.}$$

The next step here would be to differentiate both sides of the equation, but we've been through this before with the RC circuit. If you're a little more clever, there's an easier way.

Make this substitution instead $i = i_C = C\cdot\frac{d}{dt}v_C$, to get $V_{in} = R\cdot C\cdot\frac{d}{dt}v_C + L\cdot C\cdot\frac{d^2}{dt^2}v_C + v_C$

Rearrange this equation to get $V_{in} = L\cdot C\cdot\frac{d^2}{dt^2}v_C + R\cdot C\cdot\frac{d}{dt}v_C + v_C$ and $\frac{V_{in}}{L\cdot C} = \frac{d^2}{dt^2}v_C + \frac{R\cdot C}{L\cdot C}\cdot\frac{d}{dt}v_C + \frac{1}{L\cdot C}\cdot v_C$

This is the classical second-order differential equation and it is solved just like the first-order differential equation, by guessing a solution of the right form and then finding the particulars of that solution.

Standard differential equation answer: $v_C(t) = A + B\cdot e^{s\cdot t}$

Differentiate: $\frac{d}{dt}v_C = B\cdot s\cdot e^{s\cdot t}$

And again: $\frac{d^2}{dt^2}v_C = B\cdot s^2\cdot e^{s\cdot t}$

Note: It will turn out that there will be two s's (s_1 and s_2), and two B's (B and D) for the second-order solution. For now I'll leave out that added complexity.

Substitute these back into the original equation:

$$\frac{V_{in}}{L\cdot C} = \frac{d^2}{dt^2}v_C + \frac{R}{L}\cdot\frac{d}{dt}v_C + \frac{1}{L\cdot C}\cdot v_C$$

$$= B\cdot s^2\cdot e^{s\cdot t} + \frac{R}{L}\cdot B\cdot s\cdot e^{s\cdot t} + \frac{1}{L\cdot C}\cdot\left(A + B\cdot e^{s\cdot t}\right)$$

$$= B\cdot s^2\cdot e^{s\cdot t} + \frac{R}{L}\cdot B\cdot s\cdot e^{s\cdot t} + \frac{1}{L\cdot C}\cdot B\cdot e^{s\cdot t} + \frac{1}{L\cdot C}\cdot A$$

We can separate this equation into two parts, one which is time dependent and one which is not. Each part must still be an equation.

Time independent (forced) part: $V_{in} = A$, $A = V_{in} =$ final condition $= v_C(\infty)$ just like before

Time dependent (transient) part: $0 = B\cdot s^2\cdot e^{s\cdot t} + \frac{R}{L}\cdot B\cdot s\cdot e^{s\cdot t} + \frac{1}{L\cdot C}\cdot B\cdot e^{s\cdot t}$

Divide both sides by $B\cdot e^{s\cdot t}$ to get: $0 = s^2 + \frac{R}{L}\cdot s + \frac{1}{L\cdot C}$ **= characteristic equation**

This equation is important. It is called the characteristic equation and we'll need to find one like it for every second-order circuit that we analyze. Luckily, there's a much easier way to get it, using impedances similar to those we used in phasor analysis. I'll talk about that in the next section, in the meantime, let's continue with this problem.

Once you have the characteristic equation

characteristic equation: $s^2 + \frac{R}{L} \cdot s + \frac{1}{L \cdot C} = 0$

Solutions to the characteristic equation:

$$s_1 = -\frac{R}{2 \cdot L} + \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}} \qquad s_2 = -\frac{R}{2 \cdot L} - \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}}$$

This results in three possible types of solutions, depending on what's under the radical, +, -, or 0.

The characteristic equation is solved using the quadratic equation, recall:

if $a \cdot x^2 + b \cdot x + c = 0$

there are two solutions

$$x_1 = \frac{-b + \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

and

$$x_2 = \frac{-b - \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

Notice also that there are two s values (s_1 and s_2) and that leads to two two B's (we'll call them B and D)

Overdamped The part under the radical is +

if $\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C} > 0$ then s_1 and s_2 are both real and $s_1 \neq s_2$ and our guessed solution $v_C(t) = A + B \cdot e^{s \cdot t}$

will become $v_C(t) = v_C(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$ and is simply the combination of two exponentials.

Also both s_1 and s_2 will always be negative (unless you find a negative R, C, or L), meaning the exponential parts will decay with time and are thus transient.

This is the overdamped case, like a class of students on a Monday morning. Pretty dull and soon to be asleep.

Underdamped The part under the radical is -

if $\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C} < 0$ then s_1 and s_2 are both complex and and can be expressed as

$s_1 = \alpha + j \cdot \omega$ and $s_2 = \alpha - j \cdot \omega$

Well, if you start putting complex numbers in exponentials, what do you get? Euler's equations show that you'll get sines and cosines. In this case its much easier to rephrase the guessed solution like this.

$$v_C(t) = v_C(\infty) + e^{\alpha \cdot t} \cdot \left(B_2 \cdot \cos(\omega \cdot t) + D_2 \cdot \sin(\omega \cdot t)\right)$$

This form can be derived directly from $v_C(t) = A + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$

using Euler's equation, $e^{j \cdot \theta} = \cos(\theta) + j \cdot \sin(\theta)$, but we won't bother to here.

In fact, although B_2 and D_2 are <u>not</u> the same as B and D, I'll drop the "2" subscripts because we'll never actually need to convert between these two forms and the extra subscripts just become annoying.

So: $v_C(t) = v_C(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$

α and ω come from the s_1 and s_2 solutions to the characteristic equation. ω is frequency at which the underdamped circuit will "ring" or "oscillate" in response to a transient. α sets the decay rate of that oscillation. Because α will always be negative the e^{at} term insures that the transient ringing dies out in time.

This is the underdamped case, like students on spring break in Fort Lauderdale.

Natural Frequency and the Damping Ratio

These are commonly used terms to describe the underdamped response in a normalized way, similar to the τ used to decribe first-order transient responses.

The "natural frequency" is defined as: $\omega_n = \sqrt{\alpha^2 - \omega^2}$

It is the frequency that the system would oscillate at if there were no damping ($R = 0$ in our case)

for this case: $\omega_n = \frac{1}{\sqrt{L \cdot C}}$

The damping ratio is defined as: $\zeta = \frac{\alpha}{\omega_n}$ (ζ is zeta)

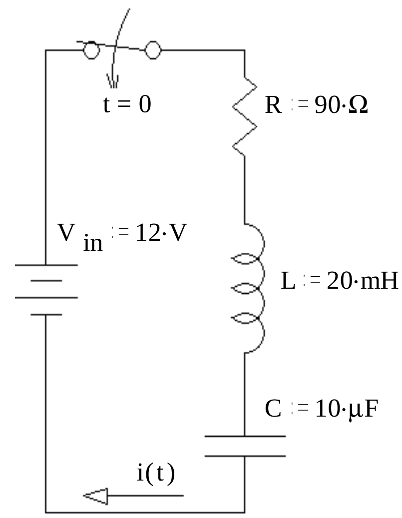
Critically damped The part under the radical is 0

if $\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C} = 0$ then s_1 and s_2 are both real and exactly the same. Now our guessed solution must be modified to $v_C(t) = v_C(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot t \cdot e^{s_2 \cdot t}$ and can result in a single overshoot.

This is actually a trivial case since it relies on an exact equality which will never happen in reality. The best use of the critically damped case is as a conceptual border between the over- and under-damped cases.

RLC examples

Let's use some component values in the RLC circuit and see what happens.



Overdamped Example

$\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C} > 0$ s_1 and s_2 are real and negative, overdamped.

$$s_1 := -\frac{R}{2 \cdot L} + \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}} \qquad s_1 = -2000 \cdot sec^{-1}$$

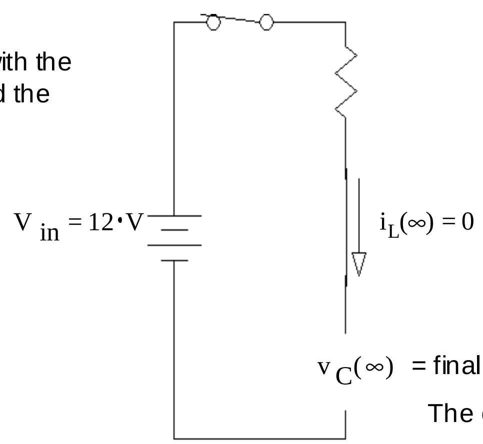
$$s_2 := -\frac{R}{2 \cdot L} - \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}} \qquad s_2 = -2500 \cdot sec^{-1}$$

$$v_C(t) = v_C(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$$

(As an example, the form is the same for all variables in this circuit)

Final conditions

REDRAW the circuit with the inductor as a short and the capacitor as an open.

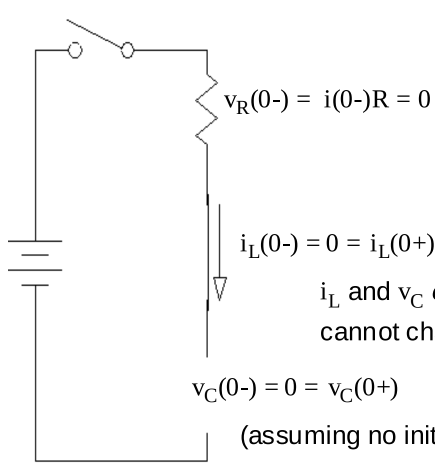


$v_C(\infty)$ = final condition = $12 \cdot V$

The capacitor will eventually charge up to V_{in}.

Initial conditions

REDRAW the circuit before the switch changes. Find two initial conditions, $i_L(0-)$ and $v_C(0-)$.

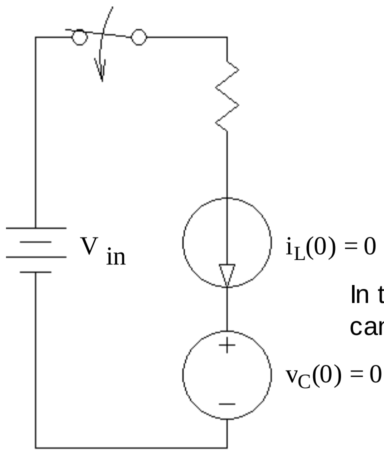


i_L and v_C ***cannot change instantly***, so cannot change the instant the switch changes.

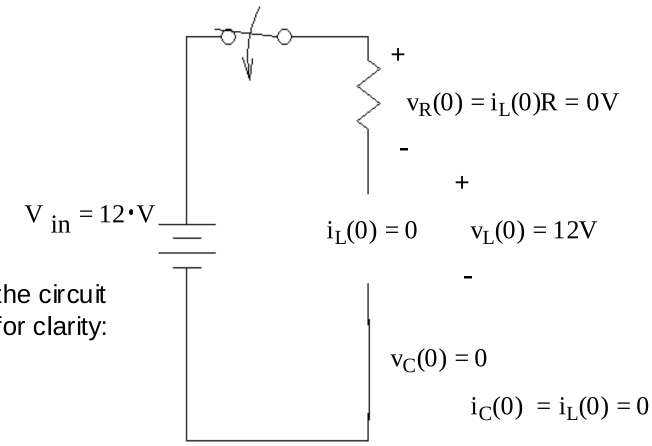
$v_C(0-) = 0 = v_C(0+)$

Pretty easy in this case (assuming no initial charge)

REDRAW the circuit again just after the switch changes. Show the inductor as a current source of $i_L(0)$ (same as $i_L(0\text{-})$) and the capacitor as a voltage source of $v_C(0)$ (same as $v_C(0\text{-})$). Find two more initial conditions, $v_L(0)$ and $i_C(0)$. Both $v_L(0)$ or $i_C(0)$ ***can change instantly***, so you ***must*** find them from $i_L(0)$ and $v_C(0)$.



In this particular case, the circuit can be redrawn again for clarity:



Again, pretty easy in this case

Rearrange the basic equations for inductors and capacitors to find the initial slopes from $v_L(0)$ or $i_C(0)$.

Rearrange $v_L = L\cdot\frac{d}{dt}i_L$ to $\frac{d}{dt}i_L(0) = \frac{v_L(0)}{L}$ $= \frac{12\cdot V}{L} = 600\cdot\frac{A}{sec}$

or, $i_C = C\cdot\frac{d}{dt}v_C$ to $\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C}$ $= \frac{0\cdot A}{C} = 0\cdot\frac{V}{sec}$

In this case

Note: You will need only the first one if you are looking for $i_L(t)$.

You will need only the second one if you are looking for $v_C(t)$.

You may need both if you are looking for any other variable in the circuit.
Other variables can usually be found most easily from $i_L(t)$ and/or $v_C(t)$.

To Find $v_C(t)$

At time $t = 0$ $v_C(0) = v_C(\infty) + B + D = 0$

$0 = 12\cdot V + B + D$ Rearranging: $D = -12\cdot V - B$

This equation has two unknowns. The initial slope will give us the needed second equation.

Differentiate the solution: $v_C(t) = v_C(\infty) + B\cdot e^{s_1\cdot t} + D\cdot e^{s_2\cdot t}$

to get: $\frac{d}{dt}v_C(t) = 0 + B\cdot s_1\cdot e^{s_1\cdot t} + D\cdot s_2\cdot e^{s_2\cdot t}$

At time $t = 0$: $\frac{d}{dt}v_C(0) = B\cdot s_1 + D\cdot s_2$

From initial conditions, above: $\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C} = 0\cdot\frac{V}{sec}$

Combining: $0\cdot\frac{V}{sec} = B\cdot s_1 + D\cdot s_2$ The second equation!

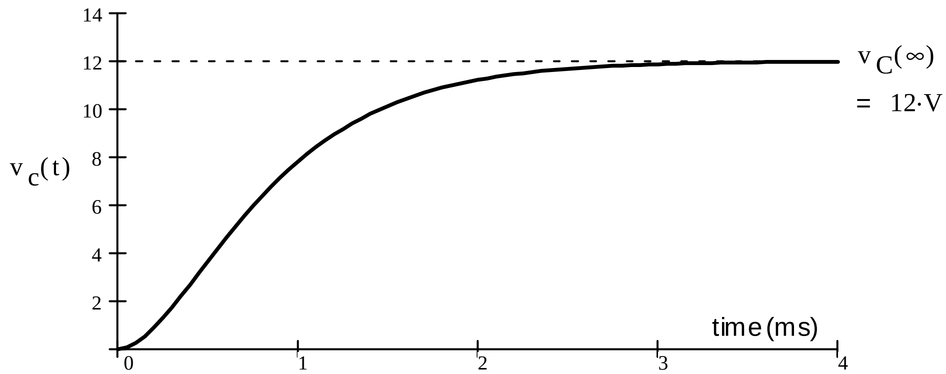
Solve simultaneously for B and D: $0\cdot\frac{V}{sec} = B\cdot s_1 + (-12\cdot V - B)\cdot s_2$ $B = s_2\cdot\frac{12\cdot V}{(s_1 - s_2)} = -60\cdot V$

$D = -12\cdot V - B = -12\cdot V - -60\cdot V = 48\cdot V$

recall the solution: $v_C(t) = v_C(\infty) + B\cdot e^{s_1\cdot t} + D\cdot e^{s_2\cdot t}$

Substitute everything back in back in: $v_C(t) := 12\cdot V - 60\cdot V\cdot e^{-\frac{2000}{sec}\cdot t} + 48\cdot V\cdot e^{-\frac{2500}{sec}\cdot t}$

Notice that this is not a simple exponential curve, although admittedly it's not much more interesting.



To Find $i_L(t)$ or $i_R(t)$ or $i_C(t)$ which all the same $i(t)$.

$$i(t) = i(\infty) + B\cdot e^{s_1\cdot t} + D\cdot e^{s_2\cdot t}$$

From final and initial conditions

$$i(0) = i(\infty) + B + D = 0 = 0 + B + D \qquad D = -B$$

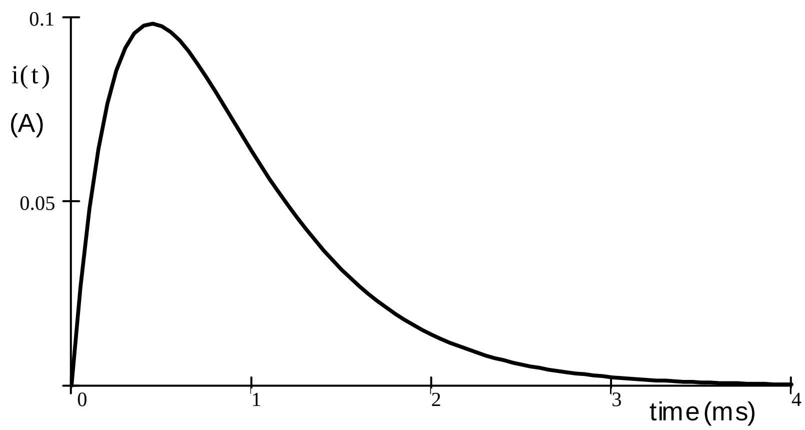
$$\frac{d}{dt}i(0) = B\cdot s_1 + D\cdot s_2 = \frac{12\cdot V}{L} = 600\cdot\frac{A}{sec}$$

Solve simultaneously for B and D $\qquad \frac{12\cdot V}{L} = B\cdot s_1 - B\cdot s_2 \qquad B = \frac{\left(\frac{12\cdot V}{L}\right)}{\left(s_1 - s_2\right)} = 1.2\cdot A$

$$D = -B = -1.2\cdot A$$

Substitute back in: $i(t) := 1.2\cdot e^{-\frac{2000}{sec}\cdot t} - 1.2\cdot e^{-\frac{2500}{sec}\cdot t}\ A$

However you get to it, at least this curve is slightly more interesting than the $v_C(t)$.



We could have found the same result from $v_C(t)$, using that to find $i_L(t)$:

$$i_C(t) = C\cdot\frac{d}{dt}v_C(t) = C\cdot\frac{d}{dt}\left(12\cdot V - 60\cdot V\cdot e^{-\frac{2000}{sec}\cdot t} + 48\cdot V\cdot e^{-\frac{2500}{sec}\cdot t}\right)$$

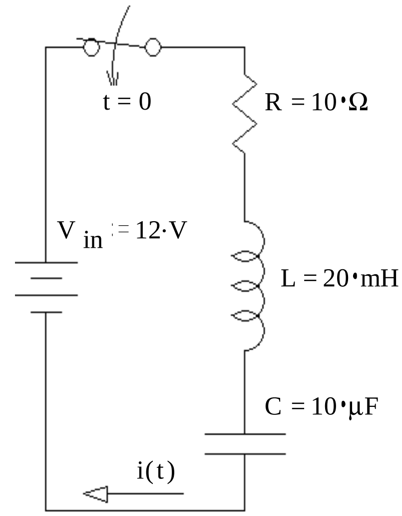
$$= C\cdot(-60\cdot V)\cdot\left(-\frac{2000}{sec}\right)\cdot e^{-\frac{2000}{sec}\cdot t} + C\cdot 48\cdot V\cdot\left(-\frac{25}{sec}\right)\cdot e^{-\frac{2500}{sec}\cdot t}$$

$C\cdot(-60\cdot V)\cdot\left(-\frac{2000}{sec}\right) = 1.2\cdot A \qquad C\cdot 48\cdot V\cdot\left(-\frac{2500}{sec}\right) = -1.2\cdot A$ and $i(t) := 1.2\cdot e^{-\frac{2000}{sec}\cdot t} - 1.2\cdot e^{-\frac{2500}{sec}\cdot t}$

same

Underdamped Example

$R := 10\cdot\Omega$ $L := 20\cdot mH$ $C := 10\cdot\mu F$



$$s_1 := -\frac{R}{2\cdot L} + \frac{1}{2}\cdot\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L\cdot C}} \qquad s_1 = -250 + 2.222\cdot 10^3 j \quad \cdot sec^{-1}$$

$$s_2 := -\frac{R}{2\cdot L} - \frac{1}{2}\cdot\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L\cdot C}} \qquad s_2 = -250 - 2.222\cdot 10^3 j \quad \cdot sec^{-1}$$

$$\alpha := -250\cdot\frac{1}{sec} \qquad \omega := Im(s_1) \qquad \omega = 2222\cdot\frac{rad}{sec}$$

The final and initial conditions are the same as before, since they did not depend on R and R is the only component that is different.

Let's find the current again this time.

$$i(t) = i(\infty) + e^{\alpha\cdot t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t))$$ (underdamped this time)

$$i(0) = i(\infty) + B,$$

$$0 = 0 + B \qquad B := 0\cdot A$$

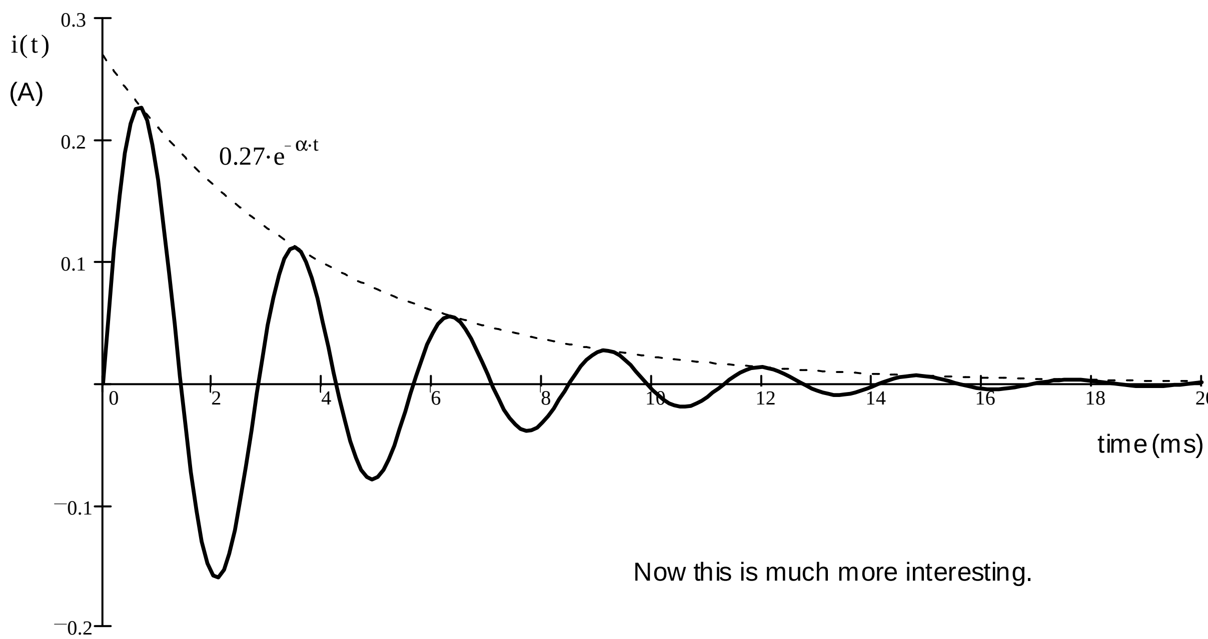
Differentiate the solution: $i(t) = i(\infty) + e^{\alpha\cdot t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t))$

to get: $$\frac{d}{dt}i(t) = \alpha\cdot e^{\alpha\cdot t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t)) + e^{\alpha\cdot t}\cdot(-B\cdot\sin(\omega\cdot t)\cdot\omega + D\cdot\cos(\omega\cdot t)\cdot\omega)$$

At time t = 0: $\frac{d}{dt}i(0) = B\cdot\alpha + D\cdot\omega$ Solve for D: $D = \dfrac{\frac{d}{dt}i(0) - B\cdot\alpha}{\omega}$

$$\frac{d}{dt}i(0) = \frac{12\cdot V}{L} \qquad D = \frac{\frac{12\cdot V}{L} - B\cdot\alpha}{\omega} = 0.27\cdot A$$

Substitute back in: $i(t) := e^{\alpha\cdot t}\cdot(0.27\cdot\sin(\omega\cdot t))$ A



Now this is much more interesting.

Critically Damped Example

First we have to figure out how to get this case

Change R's value to create critical damping: $\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C} = 0$ $\quad R := 2 \cdot \sqrt{\frac{L}{C}}$ $\quad R = 89.44271909999159 \cdot \Omega$ (exactly)

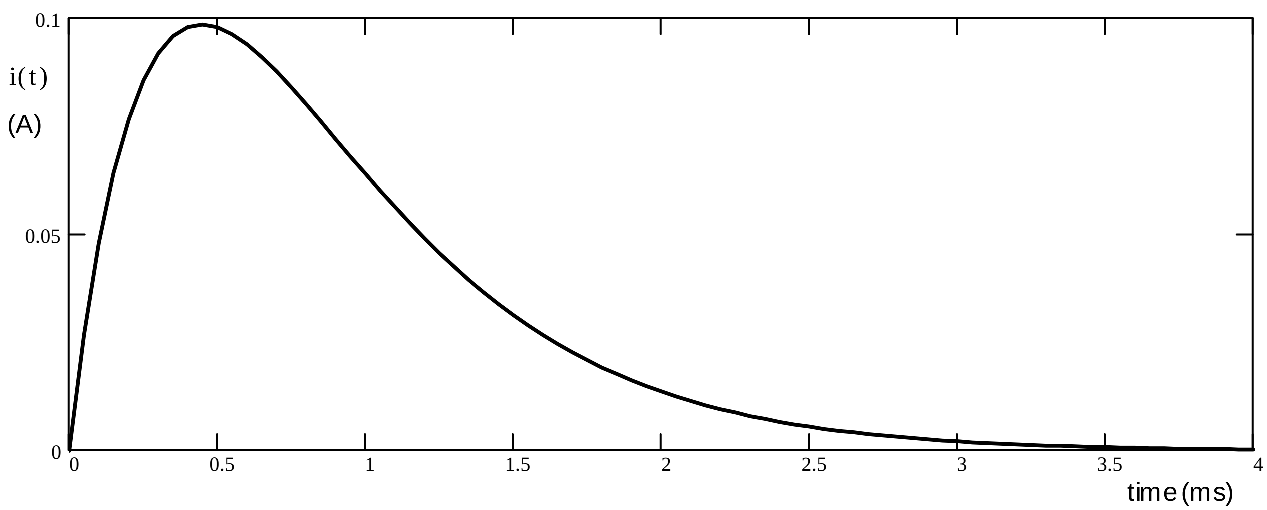
$s_1 := -\frac{R}{2 \cdot L} + \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}}$ $\quad s_1 = -2236 \cdot sec^{-1}$ $\qquad s_2 := -\frac{R}{2 \cdot L} - \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}}$ $\quad s_2 = -2236 \cdot sec^{-1}$

$i(t) = i(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot t \cdot e^{s_2 \cdot t}$ $\qquad i(\infty)$ = final condition = $0 \cdot A$ $\qquad$ Capacitor will charge up and current will stop.

$i(0) = i(\infty) + B = 0$, $B = 0$

$\frac{d}{dt} i(0) = B \cdot s_1 \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t} + D \cdot t \cdot s_2 \cdot e^{s_2 \cdot t} = B \cdot s_1 + D = \frac{12 \cdot V}{L}$ $\qquad$ Since all initial voltage will be across inductor.

Solve for D: $D = \frac{12 \cdot V}{L} = 600 \cdot \frac{A}{sec}$ $\qquad$ Substitute back in: $i(t) := 600 \cdot \frac{A}{sec} \cdot t \cdot e^{-\frac{2236}{sec} \cdot t}$



if you notice a remarkable similarity with the overdamped case, that's common for critical damping.

1.7 The Easy Way to get the Characteristic Equation

Recall from your Ordinary Differential Equations class, the Laplace transform method of solving differential equations. The Laplace transform allowed you to change time-domain functions to frequency-domain functions. We've already done this for steady-state AC circuits. We changed functions of t into functions of $j\omega$. That was the frequency domain. Laplace let's us do the same sort of thing for transients. The general procedure is as follows.

1) Transform your forcing functions into the frequency domain with the Laplace transform.

2) Solve your differential equations with plain old algebra, where:

$\frac{d}{dt}$ operation can be replaced with s, and $\int \blacksquare \, dt$ can be replaced by $\frac{1}{s}$

3) Transform your result back to the time domain with the inverse Laplace transform.

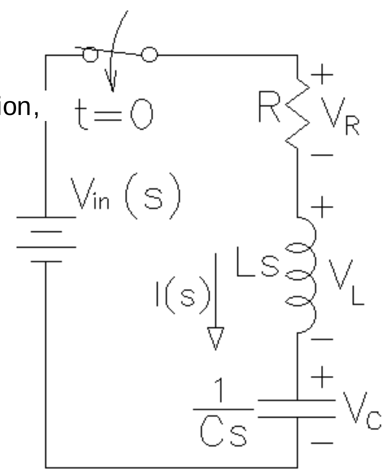
Step 1 isn't too bad, but step 3 can be a total pain without a good computer program to do the job. However, step 2 sounds great. It turns out that we can use step 2 alone and still learn a great deal about our circuits and other systems without ever bothering with steps 1 and 3.

First remember from your study of Laplace that differentiation in the time domain was the same as multiplication by s in the frequency domain. That's really all we need and we're off and running.

$$v_L(t) = L \cdot \frac{d}{dt} i_L(t) \;\; \text{--->} \;\; V_L(s) = L \cdot s \cdot I_L(s) \qquad \text{and} \qquad i_C(t) = C \cdot \frac{d}{dt} v_C(t) \;\; \text{--->} \;\; I_C(s) = C \cdot s \cdot V_C(s)$$

Leading to the Laplace impedances: Ls for an inductor and $\frac{1}{Cs}$ for a capacitor.

That's it, now we can use these impedances just like the $j\omega$ impedances, and we can use all the tools developed for DC. And with Laplace we don't even have to mess with complex numbers.



Look what happens to the RLC circuit now.

Pick any dependent variable ($I(s)$, $V_R(s)$, $V_L(s)$, or $V_C(s)$) and write a transfer function, which is a ratio of the dependent variable to the input ($V_{in}(s)$), like this:

$$V_{in}(s) = I(s) \cdot \left(\frac{1}{C \cdot s} + R + L \cdot s \right)$$

Transfer function = $H(s) = \dfrac{I(s)}{V_{in}(s)} = \dfrac{1}{\left(\frac{1}{C \cdot s} + R + L \cdot s \right)}$

Manipulate this transfer function into this form: $\dfrac{a_1 \cdot s^2 + b_1 \cdot s + k_1}{s^2 + b \cdot s + k}$

One polynomial divided by another.

$$\frac{I(s)}{V_{in}(s)} = \frac{1 \cdot (C \cdot s)}{(1 + R + L \cdot s \cdot (C \cdot s))} = \frac{\frac{1}{L} \cdot s}{\left(s^2 + \frac{R}{L} \cdot s + \frac{1}{L \cdot C} \right)}$$ in the correct form.

Set the denominator to 0 and you get the characteristic equation: $s^2 + \frac{R}{L} \cdot s + \frac{1}{L \cdot C} = 0$

At this point you just proceed with the solution like you did before; Solve the characteristic equation to find s_1 and s_2. Decide which case you have (over-, under-, or critically damped). Use the two initial conditions, $i_L(0)$ and $v_C(0)$ to find the initial condition and the initial slope of your variable of interest, then use those to find the constants B and D.

<u>Differential equation from the transfer function</u>

You can also use the transfer function to go back and find the differential equation, just replace each s with a $\frac{d}{dt}\blacksquare$ and go back to functions of t.

$$\frac{1}{L} \cdot \frac{d}{dt} V_{in}(t) = \left(\frac{d^2}{dt^2} i(t) + \frac{R}{L} \cdot \frac{d}{dt} i(t) + \frac{1}{L \cdot C} \cdot i(t) \right)$$

Actually this is a pretty useless thing to do.

ECE 2210 Second-Order Transient Examples

A.Stolp
10/29/02
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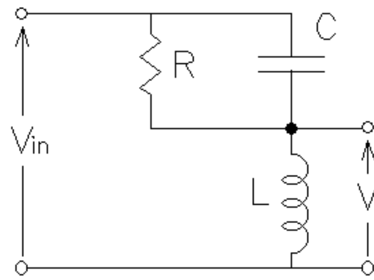
Ex. 1 a) Find the transfer function of the circuit shown. Write your equation in the form of one simple polynomial divided by another



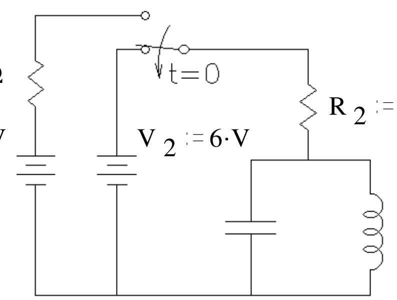
$$\mathbf{H}(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_{in}(s)} = \frac{L\cdot s}{\dfrac{1}{\dfrac{1}{R}+C\cdot s}+L\cdot s}\cdot\frac{\left(\dfrac{1}{R}+C\cdot s\right)}{\left(\dfrac{1}{R}+C\cdot s\right)}$$

$$= \frac{\dfrac{L\cdot s}{R}+L\cdot C\cdot s^2}{\left[1+L\cdot s\cdot\left(\dfrac{1}{R}+C\cdot s\right)\right]} = \frac{\dfrac{L\cdot s}{R}+L\cdot C\cdot s^2}{1+\dfrac{L\cdot s}{R}+L\cdot C\cdot s^2} = \frac{\dfrac{L\cdot s}{R}+L\cdot C\cdot s^2}{L\cdot C\cdot s^2+\dfrac{L\cdot s}{R}+1}\cdot\frac{\dfrac{1}{L\cdot C}}{\dfrac{1}{L\cdot C}} = \frac{s^2+\dfrac{1}{C\cdot R}\cdot s}{s^2+\left(\dfrac{1}{C\cdot R}\right)\cdot s+\dfrac{1}{L\cdot C}}$$

b) Find the characteristic equation $\quad 0 = s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{L\cdot C}$

Ex. 2 a) Find the characteristic equation of the circuit shown (after the switch moves to the lower position at $t = 0$).

$R_1 := 25\cdot\Omega$ $\quad V_1 := 18\cdot V$ $\quad V_2 := 6\cdot V$ $\quad R_2 := 125\cdot\Omega$



$C := 0.08\cdot\mu F \qquad L := 5\cdot mH$

$$\frac{\mathbf{V}_C(s)}{\mathbf{V}_{in}(s)} = \mathbf{H}(s) = \frac{\dfrac{1}{\dfrac{1}{L\cdot s}+C\cdot s}}{\dfrac{1}{\dfrac{1}{L\cdot s}+C\cdot s}+R_2}\cdot\frac{\left(\dfrac{1}{L\cdot s}+C\cdot s\right)}{\left(\dfrac{1}{L\cdot s}+C\cdot s\right)} = \frac{1}{1+\dfrac{R_2}{L\cdot s}+R_2\cdot C\cdot s}\cdot\frac{L\cdot s}{L\cdot s}$$

$$= \frac{L\cdot s}{L\cdot s+R_2+R_2\cdot C\cdot L\cdot s^2}\cdot\frac{\dfrac{1}{R_2\cdot C\cdot L}}{\dfrac{1}{R_2\cdot C\cdot L}} = \frac{\dfrac{1}{R_2\cdot C}\cdot s}{\dfrac{1}{R_2\cdot C}\cdot s+\dfrac{1}{L\cdot C}+s^2}$$

Characteristic equation:

$$0 = s^2+\frac{1}{R_2\cdot C}\cdot s+\frac{1}{L\cdot C}$$

b) Find the solutions of the characteristic equation. $\quad \dfrac{1}{R_2\cdot C} = 1\cdot 10^5\ \cdot sec^{-1} \qquad \dfrac{4}{L\cdot C} = 1\cdot 10^{10}\ \cdot sec^{-2}$

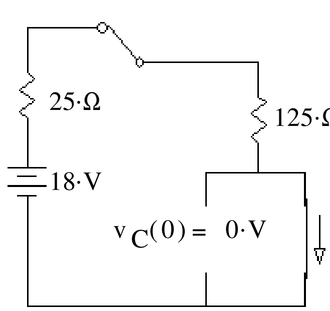
$$s_1 := \frac{-\dfrac{1}{R_2\cdot C}+\sqrt{\left(\dfrac{1}{R_2\cdot C}\right)^2-\dfrac{4}{L\cdot C}}}{2} \qquad s_1 = -5\cdot 10^4\ \cdot\frac{1}{sec}$$

$$s_2 := \frac{-\dfrac{1}{R_2\cdot C}-\sqrt{\left(\dfrac{1}{R_2\cdot C}\right)^2-\dfrac{4}{L\cdot C}}}{2} \qquad s_2 = -5\cdot 10^4\ \cdot\frac{1}{sec}$$

$s_1 = s_2$ so... critically damped

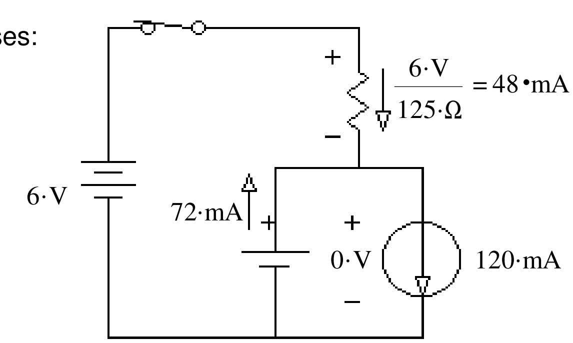
c) Find initial and final conditions for $v_C(t)$

before switch is moved:



$$i_L(0) = \frac{18\cdot V}{25\cdot\Omega+125\cdot\Omega} = 120\cdot mA$$

just after switch closes:

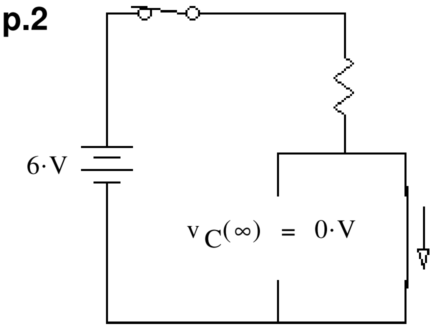


$$\frac{6\cdot V}{125\cdot\Omega} = 48\cdot mA$$

$$\frac{d}{dt}v_C(0) = \frac{-72\cdot mA}{C} = -9\cdot 10^5\ \cdot\frac{V}{sec}$$

Second-Order Transient Examples, p.2

Final conditions:



$v_C(\infty) = 0\cdot V$ $\qquad i_L(\infty) = \dfrac{6\cdot V}{125\cdot\Omega} = 48\cdot mA$

d) Find the full expression of $v_C(t)$.

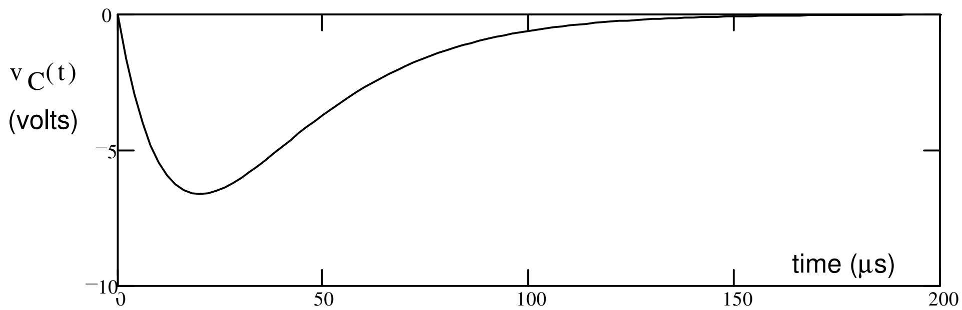
Critically damped $\qquad v_C(t) = v_C(\infty) + B\cdot e^{s_1\cdot t} + D\cdot t\cdot e^{s_2\cdot t}$

$v_C(0) = v_C(\infty) + B$

$0 = 0 + B \qquad B = 0 \qquad \dfrac{d}{dt}v_C(0) = B\cdot s + D$

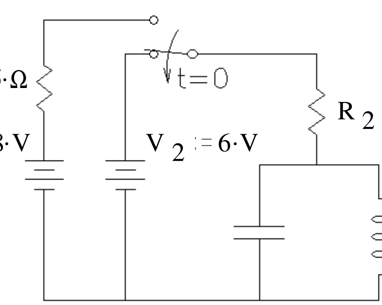
$-9\cdot 10^5\cdot\dfrac{V}{sec} = 0\cdot\left(-5\cdot 10^4\cdot\dfrac{1}{sec}\right) + D \qquad D := -9\cdot 10^5\cdot\dfrac{V}{sec}$

$v_C(t) := -9\cdot 10^5\cdot\dfrac{V}{sec}\cdot t\cdot e^{s_1\cdot t}$



Ex. 2 with bigger R_2

a) Find the characteristic equation of the circuit shown (after the switch moves to the lower position at t = 0).
as before:



$R_1 := 25\cdot\Omega \qquad R_2 := 1\cdot k\Omega \qquad V_1 := 18\cdot V \qquad V_2 := 6\cdot V$

$C := 0.08\cdot\mu F \qquad L := 5\cdot mH$

$$\frac{\mathbf{V}_C(s)}{\mathbf{V}_{in}(s)} = \mathbf{H}(s) = \frac{\dfrac{1}{R_2\cdot C}\cdot s}{\dfrac{1}{R_2\cdot C}\cdot s + \dfrac{1}{L\cdot C} + s^2}$$

Characteristic equation:

$$0 = s^2 + \frac{1}{R_2\cdot C}\cdot s + \frac{1}{L\cdot C}$$

b) Find the solutions of the characteristic equation. $\qquad \dfrac{1}{R_2\cdot C} = 1.25\cdot 10^4 \cdot sec^{-1} \qquad \dfrac{4}{L\cdot C} = 1\cdot 10^{10}\cdot sec^{-2}$

$$s_1 := \frac{-\dfrac{1}{R_2\cdot C} + \sqrt{\left(\dfrac{1}{R_2\cdot C}\right)^2 - \dfrac{4}{L\cdot C}}}{2}$$

$s_1 = -6.25\cdot 10^3 + 4.961\cdot 10^4 j \quad \cdot\dfrac{1}{sec}$

$s_2 = \overline{s_1} = -6.25\cdot 10^3 - 4.961\cdot 10^4 j \quad \cdot\dfrac{1}{sec}$

s_1 & s_2 complex, so underdamped

$\alpha := Re(s_1) \qquad \alpha = -6.25\cdot 10^3 \cdot sec^{-1} \qquad \omega := Im(s_1) \qquad \omega = 4.961\cdot 10^4 \cdot sec^{-1}$

c) Find initial and final conditions for $v_C(t)$
See drawings above

$$v_C(0) = 0\cdot V$$

$$i_L(0) = \frac{18\cdot V}{25\cdot\Omega + 1\cdot k\Omega} = 17.561\cdot mA$$

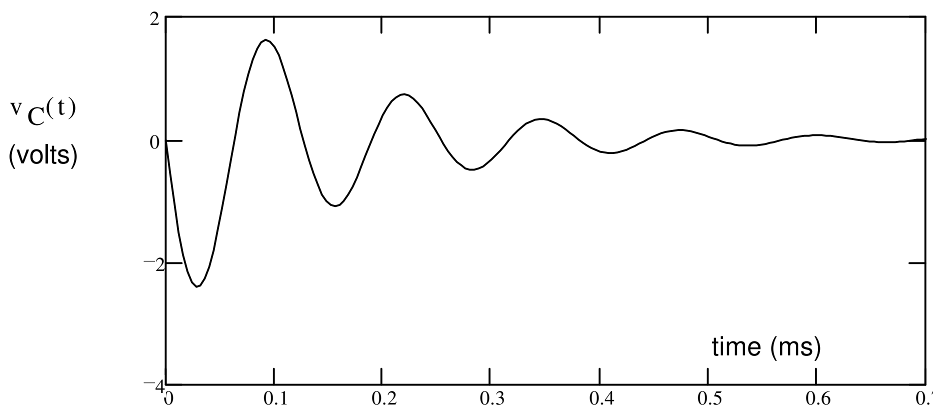
$$i_C(0) = \frac{6\cdot V}{1\cdot k\Omega} - 17.561\cdot mA = -11.561\cdot mA \qquad \frac{d}{dt}v_C(0) = \frac{-11.561\cdot mA}{C} = -1.445\cdot 10^5\cdot\frac{V}{sec}$$

$$v_C(\infty) = 0\cdot V$$

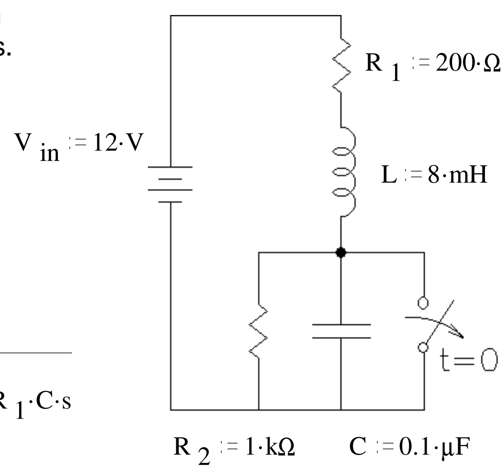
d) Find the full expression of $v_C(t)$. <u>Underdamped</u> $v_C(t) = v_C(\infty) + e^{\alpha t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t))$

$$B = v_C(0) - v_C(\infty) = 0\cdot V - 0\cdot V = 0\cdot V$$

$$D = \frac{\frac{d}{dt}v_C(0) - B\cdot\alpha}{\omega} = \frac{-1.445\cdot 10^5\cdot\frac{V}{sec} - 0\cdot V\cdot\alpha}{\omega} = -2.913\cdot V \qquad v_C(t) := -2.913\cdot V\cdot e^{-\frac{6250}{sec}\cdot t}\cdot\sin\left(\frac{49610}{sec}\cdot t\right)$$



Ex. 3 a) Find the characteristic equation of the circuit shown. (after the switch opens at t = 0). Write your equation in the form of a simple polynomials.



$$\mathbf{H}(s) = \frac{\mathbf{I_T}(s)}{\mathbf{V_{in}}(s)} = \frac{1}{\mathbf{Z}(s)} = \frac{1}{\frac{1}{\frac{1}{R_2} + C\cdot s} + L\cdot s + R_1}\cdot\frac{\left(\frac{1}{R_2} + C\cdot s\right)}{\left(\frac{1}{R_2} + C\cdot s\right)}$$

$$= \frac{\frac{1}{R_2} + C\cdot s}{1 + L\cdot s\cdot\left(\frac{1}{R_2} + C\cdot s\right) + R_1\cdot\left(\frac{1}{R_2} + C\cdot s\right)} = \frac{\frac{1}{R_2} + C\cdot s}{1 + \frac{L\cdot s}{R_2} + L\cdot C\cdot s^2 + \frac{R_1}{R_2} + R_1\cdot C\cdot s}$$

$$= \frac{\frac{1}{R_2} + C\cdot s}{L\cdot C\cdot s^2 + \frac{L\cdot s}{R_2} + R_1\cdot C\cdot s + 1 + \frac{R_1}{R_2}}\cdot\frac{\frac{1}{L\cdot C}}{\frac{1}{L\cdot C}} = \frac{\frac{1}{L\cdot C\cdot R_2} + \frac{C}{L\cdot C}\cdot s}{s^2 + \left(\frac{L}{L\cdot C\cdot R_2} + \frac{R_1\cdot C}{L\cdot C}\right)\cdot s + \left(1 + \frac{R_1}{R_2}\right)\cdot\frac{1}{L\cdot C}}$$

$$= \frac{1}{L}\cdot\frac{s + \frac{1}{C\cdot R_2}}{s^2 + \left(\frac{1}{C\cdot R_2} + \frac{R_1}{L}\right)\cdot s + \left(1 + \frac{R_1}{R_2}\right)\cdot\frac{1}{L\cdot C}}$$

Characteristic eq.: $0 = s^2 + \left(\frac{1}{C\cdot R_2} + \frac{R_1}{L}\right)\cdot s + \left(1 + \frac{R_1}{R_2}\right)\cdot\frac{1}{L\cdot C}$

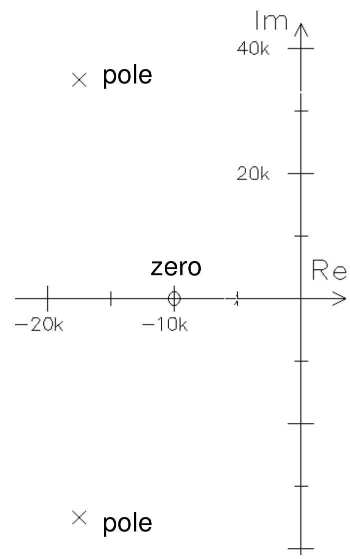
b) Find the solutions (numbers) of the characteristic equation:

$b := \frac{1}{C \cdot R_2} + \frac{R_1}{L}$ $\quad b = 3.5 \cdot 10^4 \cdot \mathrm{sec}^{-1}$ $\qquad k := \left(1 + \frac{R_1}{R_2}\right) \cdot \frac{1}{L \cdot C}$ $\quad k = 1.5 \cdot 10^9 \cdot \mathrm{sec}^{-2}$

$s_1 := \frac{-b + \sqrt{b^2 - 4 \cdot k}}{2}$ $\quad s_1 = -1.75 \cdot 10^4 + 3.455 \cdot 10^4 j \cdot \frac{1}{\mathrm{sec}}$ $\qquad \alpha := -\frac{b}{2}$ $\quad \alpha = -1.75 \cdot 10^4 \cdot \mathrm{sec}^{-1}$

$s_2 := \frac{-b - \sqrt{b^2 - 4 \cdot k}}{2}$ $\quad s_2 = -1.75 \cdot 10^4 - 3.455 \cdot 10^4 j \cdot \frac{1}{\mathrm{sec}}$ $\qquad \omega := \frac{1}{2} \cdot \sqrt{4 \cdot k - b^2}$ $\quad \omega = 3.455 \cdot 10^4 \cdot \mathrm{sec}^{-1}$

Underdamped



c) Plot the poles and zeroes of the transfer function.

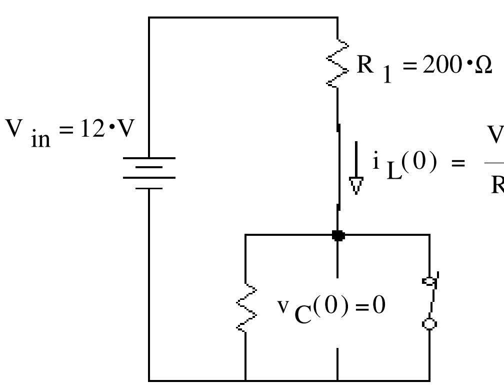
The poles are the s's where the denominator is zero, that is, the s_1 & s_2 solutions to the characteristic equation.

The zero is the s where the numerator is zero: $0 = \frac{1}{L \cdot C \cdot R_2} + \frac{C}{L \cdot C} \cdot s$

$s = -\frac{1}{C \cdot R_2} = -1 \cdot 10^4 \cdot \mathrm{sec}^{-1}$

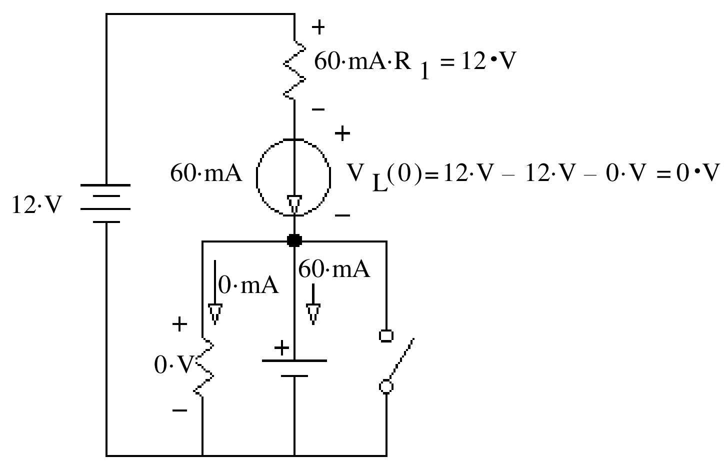
d) Initial and final conditions for $i_L(t)$ and $v_C(t)$.

before the switch opens



$i_L(0) = \frac{V_{in}}{R_1} = 60 \cdot mA$

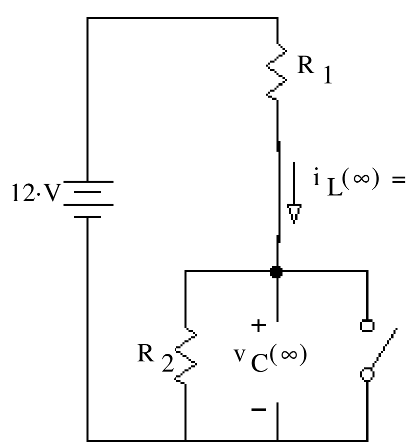
just after the switch opens



$\frac{d}{dt} i_L(0) = \frac{V_L(0)}{L} = \frac{0 \cdot V}{L} = 0 \cdot \frac{A}{\mathrm{sec}}$

$\frac{d}{dt} v_C(0) = \frac{i(0)}{C} = \frac{60 \cdot mA}{C} = 6 \cdot 10^5 \cdot \frac{V}{\mathrm{sec}}$

Final condition:



$i_L(\infty) = \frac{V_{in}}{R_1 + R_2} = 10 \cdot mA$

$v_C(\infty) = \frac{R_2}{R_1 + R_2} \cdot V_{in} = \frac{1 \cdot k\Omega}{200 \cdot \Omega + 1 \cdot k\Omega} \cdot 12 \cdot V = 10 \cdot V$

e) Find the full expression of $i_L(t)$.

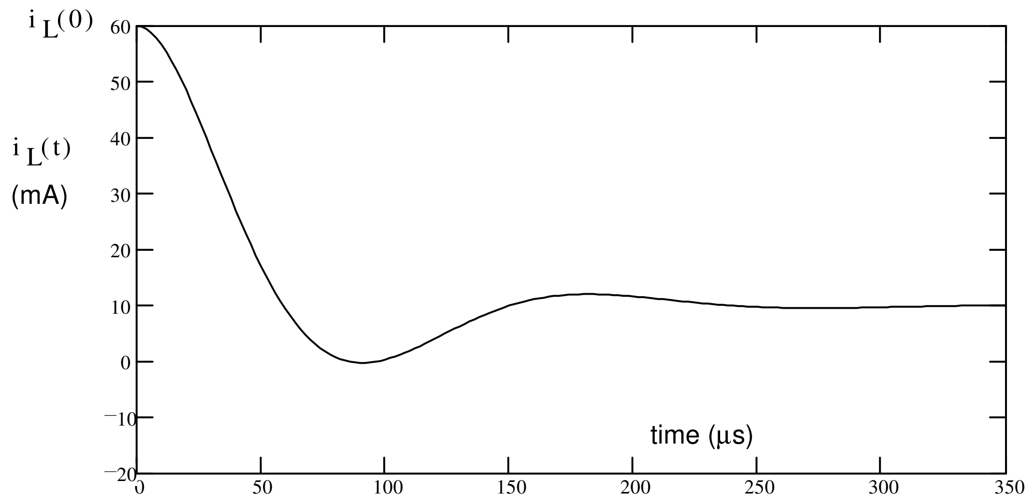
Underdamped $X(t) = X(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$

$i_L(t) = i_L(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$

$i_L(0) = i_L(\infty) + B$ so.. $B = i_L(0) - i_L(\infty)$ $B := 60 \cdot mA - 10 \cdot mA$ $B = 50 \cdot mA$

$\frac{d}{dt} i_L(0) = B \cdot \alpha + D \cdot \omega$ so.. $D = \dfrac{\frac{d}{dt} i_L(0) - B \cdot \alpha}{\omega}$ $D := \dfrac{0 \cdot \frac{A}{sec} - B \cdot \alpha}{\omega}$ $D = 25.325 \cdot mA$

$$i_L(t) := 10 \cdot mA + e^{-\frac{17500}{sec} \cdot t} \cdot \left(50 \cdot mA \cdot \cos\left(\frac{34550}{sec} \cdot t\right) + 25.325 \cdot mA \cdot \sin\left(\frac{34550}{sec} \cdot t\right)\right)$$



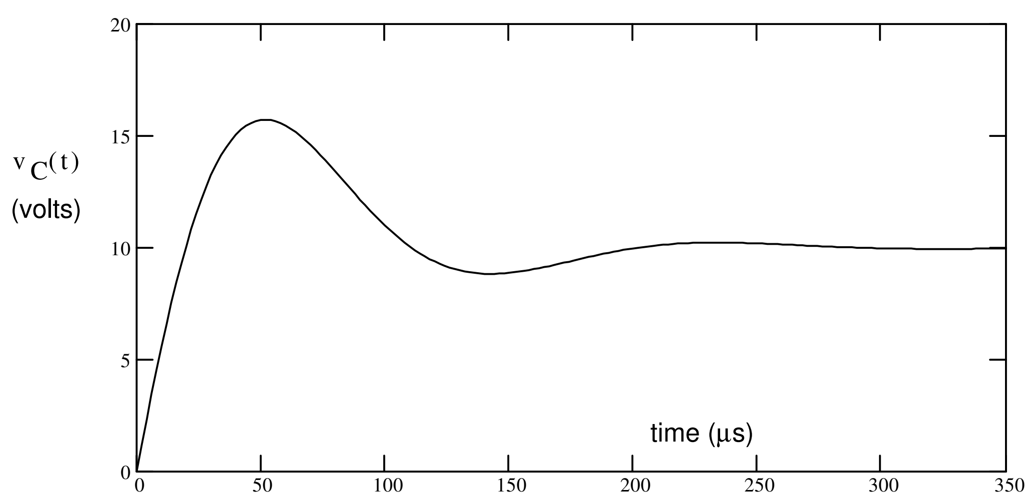
$i_L(\infty) = 10 \cdot mA$

f) Find the full expression of $v_C(t)$.

$B := 0 \cdot V - 10 \cdot V$ $B = -10 \cdot V$ $D := \dfrac{6 \cdot 10^5 \cdot \frac{V}{sec} - B \cdot \alpha}{\omega}$ $D = 12.301 \cdot V$

$v_C(t) = v_C(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$

$$v_C(t) := 10 \cdot V + e^{-\frac{17500}{sec} \cdot t} \cdot \left(-10 \cdot V \cdot \cos\left(\frac{34550}{sec} \cdot t\right) + 12.301 \cdot V \cdot \sin\left(\frac{34550}{sec} \cdot t\right)\right)$$



$v_C(\infty) = 10 \cdot V$

h) What value of R_1 would make this system critically damped?

$$\left(\frac{1}{C\cdot R_2}+\frac{R_1}{L}\right)^2 = 4\cdot\left(1+\frac{R_1}{R_2}\right)\cdot\frac{1}{L\cdot C} \qquad \frac{1}{C^2\cdot R_2^{\,2}}+\frac{2}{C\cdot R_2}\cdot\frac{R_1}{L}+\frac{R_1^{\,2}}{L^2} = \frac{4}{L\cdot C}+\frac{4}{C\cdot R_2}\cdot\frac{R_1}{L}$$

$$0 = \left(\frac{1}{L^2}\cdot R_1^{\,2}-\frac{2}{L\cdot C\cdot R_2}\cdot R_1\right)+\left(\frac{1}{C^2\cdot R_2^{\,2}}-\frac{4}{L\cdot C}\right) \qquad 0 = \left(R_1^{\,2}-\frac{2\cdot L}{C\cdot R_2}\cdot R_1\right)+\left(\frac{L^2}{C^2\cdot R_2^{\,2}}-\frac{4\cdot L}{C}\right)$$

Solve for R_1 with quadradic equation:

$$R_1 = \frac{\frac{2\cdot L}{C\cdot R_2}+\sqrt{\left(\frac{2\cdot L}{C\cdot R_2}\right)^2-4\cdot\left(\frac{L^2}{C^2\cdot R_2^{\,2}}-\frac{4\cdot L}{C}\right)}}{2}$$

Quadradic equation can be reduced to:

$$= \frac{L}{C\cdot R_2}-\frac{4}{2}\cdot\sqrt{\frac{L}{C}} = -485.7\cdot\Omega$$ this solution can't be

$$= \frac{L}{C\cdot R_2}+\frac{4}{2}\cdot\sqrt{\frac{L}{C}} = 645.7\cdot\Omega$$ this must be the solution

Ex. 2 with bigger R_1

$R_1 := 1\cdot k\Omega$ This should make the system overdamped

$$b := \frac{1}{C\cdot R_2}+\frac{R_1}{L} \qquad b = 1.35\cdot 10^5\ \cdot sec^{-1} \qquad k := \left(1+\frac{R_1}{R_2}\right)\cdot\frac{1}{L\cdot C} \qquad k = 2.5\cdot 10^9\ \cdot sec^{-2}$$

$$s_1 := \frac{-b+\sqrt{b^2-4\cdot k}}{2} \qquad s_1 = -2.215\cdot 10^4\ \cdot\frac{1}{sec} \qquad s_2 := \frac{-b-\sqrt{b^2-4\cdot k}}{2} \qquad s_2 = -1.128\cdot 10^5\ \cdot\frac{1}{sec}$$

Overdamped

$$v_C(0) = 0 \qquad i_L(0) = \frac{V_{in}}{R_1} = 12\cdot mA = i_C(0) \qquad \frac{d}{dt}v_C(0) = \frac{i(0)}{C} = \frac{12\cdot mA}{C} = 1.2\cdot 10^5\ \cdot\frac{V}{sec}$$

$$v_C(\infty) = \frac{R_2}{R_1+R_2}\cdot V_{in} = 6\cdot V \qquad i_L(\infty) = \frac{V_{in}}{R_1+R_2} = 6\cdot mA$$

$$1.2\cdot 10^5\cdot\frac{1}{sec} =$$

$$v_C(0) = v_C(\infty)+B+D$$

$$0\cdot V = 6\cdot V+B+D \qquad B = -(6\cdot V+D)$$

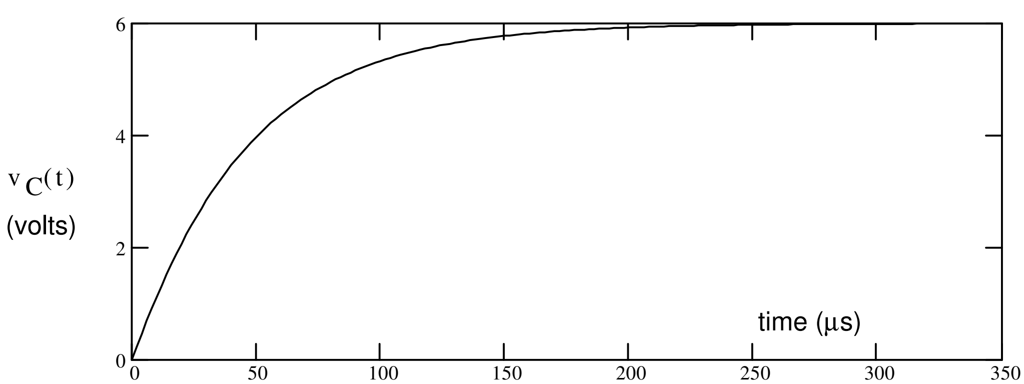
$$\frac{d}{dt}v_C(0) = B\cdot s_1+D\cdot s_2 = -6\cdot V\cdot s_1-D\cdot s_1+D\cdot s_2 \qquad D := \frac{1.2\cdot 10^5\cdot\frac{V}{sec}+6\cdot V\cdot s_1}{s_2-s_1}$$

$$D = 0.143\cdot V$$

$$B = -(6\cdot V+D) = -6.143\cdot V$$

$$v_C(t) = v_C(\infty)+B\cdot e^{s_1\cdot t}+D\cdot e^{s_2\cdot t} \qquad v_C(t) := 6\cdot V-6.143\cdot V\cdot e^{-\frac{2.215\cdot 10^4}{sec}\cdot t}+0.143\cdot V\cdot e^{-\frac{1.128\cdot 10^5}{sec}\cdot t}$$

$$v_C(\infty) = 6\cdot V$$

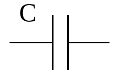


Notes, Second Order Transients

ECE 2210

A.Stolp
4/6/00,
2/25/16,
10/23/23

Laplace impedances

Resistor	Capacitor	Inductor
 R	C 	 L
$Z_R = R$	$Z_C = \frac{1}{C \cdot s}$	$Z_L = L \cdot s$

Transfer function

Use Laplace impedances, manipulate your circuit equation(s) to find a transfer function:

May be I_X or any desired variable

a_1, b_1, k_1 coefficients may be zero

Rearrange circuit equation to: $\mathbf{H}(s) = \frac{\text{output}}{\text{input}} = \frac{\mathbf{V}_X(s)}{\mathbf{V}_{in}(s)} = \frac{a_1 \cdot s^2 + b_1 \cdot s + k_1}{s^2 + b \cdot s + k}$ = transfer function

May be I_{in} or any forcing function

Characteristic equation

$$s^2 + b \cdot s + k = 0$$

To find the poles of the transfer function — characteristic equation

Complete solution

Solutions to the characteristic equation: $s_1 = -\frac{b}{2} + \frac{\sqrt{b^2 - 4 \cdot k}}{2}$ $\quad s_2 = -\frac{b}{2} - \frac{\sqrt{b^2 - 4 \cdot k}}{2}$

Find initial Conditions (v_C and/or i_L)

Find conditions of just before time $t = 0$, $v_C(0-)$ and $i_L(0-)$. These will be the same just after time $t = 0$, $v_C(0+)$ and $i_L(0+)$ and will be your initial conditions.

Use normal circuit analysis to find your desired variable: $v_X(0)$ or $i_X(0)$

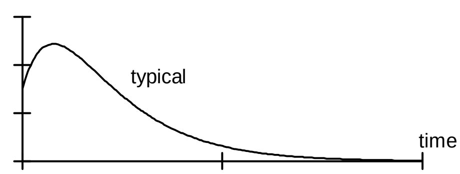
Also find: $\frac{d}{dt} v_X(0)$ or $\frac{d}{dt} i_X(0)$ The trick to finding these is to see that: $\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C}$ and $\frac{d}{dt} i_L(0) = \frac{v_L(0)}{L}$

Find final conditions ("steady-state" or "forced" solution)

DC inputs: Inductors are shorts Capacitors are opens Solve by DC analysis $v_X(\infty)$ or $i_X(\infty)$

AC inputs: Solve by AC steady-state analysis using $j\omega$

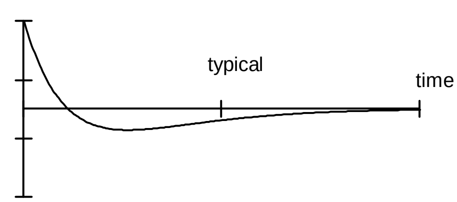
X(t) may be replaced by $v_X(t)$, $i_X(t)$ or any desired variable in the equations below



Overdamped $\quad b^2 - 4 \cdot k > 0 \quad$ s_1 and s_2 are real and negative

$$X(t) = X(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$$

$X(0) = X(\infty) + B + D \qquad \frac{d}{dt} X(0) = B \cdot s_1 + D \cdot s_2$ Solve simultaneously for B and D.



Critically damped $\quad b^2 - 4 \cdot k = 0 \quad s_1 = s_2 = -\frac{b}{2} = s \quad$ s_1 and s_2 are real, equal and negative

$$X(t) = X(\infty) + B \cdot e^{s \cdot t} + D \cdot t \cdot e^{s \cdot t}$$

$X(0) = X(\infty) + B$

so.. $B = X(0) - X(\infty) \qquad \frac{d}{dt} X(0) = B \cdot s + D \qquad$ so.. $D = \frac{d}{dt} X(0) - B \cdot s$

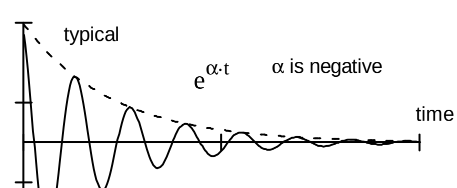
Underdamped $\quad b^2 - 4 \cdot k < 0 \quad s_1 = \alpha + j \cdot \omega \quad s_2 = \alpha - j \cdot \omega \quad \alpha$ is negative

complex s_1 and s_2

$$X(t) = X(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$$

$X(0) = X(\infty) + B \qquad \frac{d}{dt} X(0) = B \cdot \alpha + D \cdot \omega \quad$ so.. $D = \frac{\frac{d}{dt} X(0) - B \cdot \alpha}{\omega}$

so.. $B = X(0) - X(\infty)$



$f = \frac{\omega}{2 \cdot \pi}$

Derivation of the Canned Solutions, Second Order Transients

ECE 2210

A.Stolp
10/23/23

How do we find B and D ?? You will use the canned solutions, which I will derive here, using initial conditions.

These are worked out within an example, starting on page 1.12 of the main Second-Order Transients handout.

Overdamped

Let's assume we've found that s_1 and s_2 are real and negative, and you're interested in the capacitor voltage.

Then: $v_C(t) = v_C(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$

At time $t = 0$ $v_C(0) = v_C(\infty) + B + D = v_C(0-)$, whatever it was just before time t = 0. It CANNOT change instantly
Same for $i_L(0)$

But that's only one equation, and we have two unknowns, B and D.

The trick is to differentiate the solution: $v_C(t) = v_C(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$

$$\frac{d}{dt} v_C(t) = 0 + B \cdot s_1 \cdot e^{s_1 \cdot t} + D \cdot s_2 \cdot e^{s_2 \cdot t}$$

At time t = 0: $\frac{d}{dt} v_C(0) = B \cdot s_1 + D \cdot s_2$ = initial slope

From initial conditions, above: $\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C} = B \cdot s_1 + D \cdot s_2$ The second equation !

Solve simultaneously for B and D.

But i_C CAN change instantly, so...

We will find $i_C(0)$ from $i_L(0) = i_L(0-)$ because i_L can't change instantly

This will require circuit analysis at time t = 0+

Underdamped

Let's assume we've found complex s_1 and s_2 $s_1 = \alpha + j \cdot \omega$ $s_2 = \alpha - j \cdot \omega$ α is negative

Then: $v_C(t) = v_C(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$

At time $t = 0$ $v_C(0) = v_C(\infty) + B = v_C(0-)$ $B = v_C(0) - v_C(\infty)$

Now differentiate the solution: $v_C(t) = v_C(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$

recall: $\frac{d}{dt}(f(t) \cdot g(t)) = \left(\frac{d}{dt} f(t)\right) \cdot g(t) + f(t) \cdot \left(\frac{d}{dt} g(t)\right)$

yields: $\frac{d}{dt} v_C(t) = \alpha \cdot e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t)) + e^{\alpha \cdot t} \cdot (-B \cdot \sin(\omega \cdot t) \cdot \omega + D \cdot \cos(\omega \cdot t) \cdot \omega)$

At time t = 0: $\frac{d}{dt} v_C(0) = B \cdot \alpha + D \cdot \omega$ Solve for: $D = \dfrac{\frac{d}{dt} v_C(0) - B \cdot \alpha}{\omega}$

Critically damped

Let's assume we've found real $s_1 = s_2 = s$

Then: $v_C(t) = v_C(\infty) + B \cdot e^{s \cdot t} + D \cdot t \cdot e^{s \cdot t}$

At time $t = 0$ $v_C(0) = v_C(\infty) + B = v_C(0-)$ $B = v_C(0) - v_C(\infty)$

Now differentiate the solution: $\frac{d}{dt} v_C(t) = B \cdot s \cdot e^{s \cdot t} + D \cdot e^{s \cdot t} + D \cdot t \cdot s \cdot e^{s \cdot t}$

$\frac{d}{dt} v_C(0) = B \cdot s + D$ Solve for: $D = \frac{d}{dt} v_C(0) - B \cdot s$

Same goes for and variable (like $i_L(t)$, for example). $v_C(0+) = v_C(0-)$ $i_L(0+) = i_L(0-)$

$\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C}$ $\frac{d}{dt} i_L(0) = \frac{v_L(0)}{L}$ And circuit analysis at time t = 0+