ECE 2210 Lecture 18 notes Second order Transient examples

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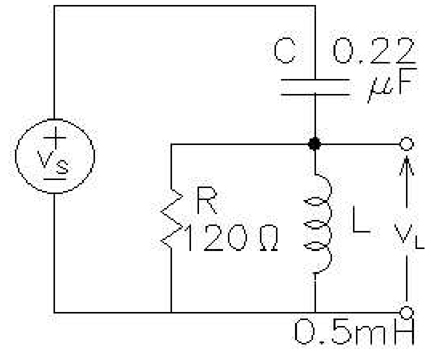
10/30/06
2/19/10

Ex. 1 For the circuit shown:



a) Find the transfer function v_L.

$$\mathbf{V_L}(s) = \frac{\dfrac{1}{\dfrac{1}{L\cdot s}+\dfrac{1}{R}}}{\dfrac{1}{\dfrac{1}{L\cdot s}+\dfrac{1}{R}}+\dfrac{1}{C\cdot s}}\cdot\mathbf{V_S}(s)$$

$$= \frac{1}{1+\dfrac{1}{C\cdot s}\cdot\left(\dfrac{1}{L\cdot s}+\dfrac{1}{R}\right)}\cdot\mathbf{V_S}(s) \quad = \frac{1}{1+\dfrac{1}{C\cdot s}\cdot\dfrac{1}{L\cdot s}+\dfrac{1}{C\cdot s}\cdot\dfrac{1}{R}}\cdot\mathbf{V_S}(s) \quad = \frac{s^2}{s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{L\cdot C}}\cdot\mathbf{V_S}(s)$$

$$\mathbf{H}(s) = \frac{\mathbf{V_L}(s)}{\mathbf{V_S}(s)} = \frac{s^2}{s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{L\cdot C}}$$

$$= \frac{s^2}{s^2+\dfrac{3.788\cdot 10^4}{sec}\cdot s+\dfrac{9.091\cdot 10^9}{sec^2}}$$

$R := 120\cdot\Omega \qquad C := 0.22\cdot\mu F \qquad L := 0.5\cdot mH$

$$\frac{1}{C\cdot R} = 3.788\bullet 10^4 \bullet\frac{1}{sec} \qquad \frac{1}{L\cdot C} = 9.091\bullet 10^9 \bullet\frac{1}{sec^2}$$

b) Find the characteristic equation for this circuit.

$$0 = s^2+\frac{1}{C\cdot R}\cdot s+\frac{1}{L\cdot C} = s^2+\frac{3.788\cdot 10^4}{sec}\cdot s+\frac{9.091\cdot 10^9}{sec^2}$$

Just the denominator set to zero. The solutions of the characteristic equation are the "poles" of the transfer function.

c) Find the differential equation for v_L.

Cross-multiply the transfer function

$$s^2\cdot\mathbf{V_S}(s) = \left(s^2+\frac{1}{C\cdot R}\cdot s+\frac{1}{L\cdot C}\right)\cdot\mathbf{V_L}(s)$$

$$s^2\cdot\mathbf{V_S}(s) = s^2\cdot\mathbf{V_L}(s)+\frac{1}{C\cdot R}\cdot s\cdot\mathbf{V_L}(s)+\frac{1}{L\cdot C}\cdot\mathbf{V_L}(s)$$

$$\frac{d^2}{dt^2}v_S(t) = \frac{d^2}{dt^2}v_L(t)+\frac{1}{C\cdot R}\cdot\frac{d}{dt}v_L(t)+\frac{1}{L\cdot C}\cdot v_L(t)$$

$$\frac{d^2}{dt^2}v_S(t) = \frac{d^2}{dt^2}v_L(t)+\frac{3.788\cdot 10^4}{sec}\cdot\frac{d}{dt}v_L(t)+\frac{9.091\cdot 10^9}{sec^2}\cdot v_L(t)$$

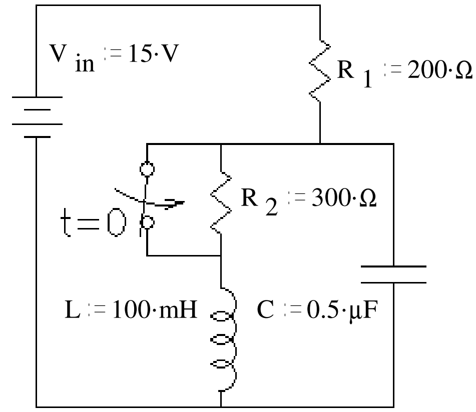
d) What are the solutions to the characteristic equation?

$$s_1 = \frac{-3.788\cdot 10^4}{2}+\frac{1}{2}\cdot\sqrt{\left(3.788\cdot 10^4\right)^2-4\cdot\left(9.091\cdot 10^9\right)} = -1.894\bullet 10^4 + 9.345\bullet 10^4 j$$

$$s_2 = \frac{-3.788\cdot 10^4}{2}-\frac{1}{2}\cdot\sqrt{\left(3.788\cdot 10^4\right)^2-4\cdot\left(9.091\cdot 10^9\right)} = -1.894\bullet 10^4 - 9.345\bullet 10^4 j$$

e) What type of response do you expect from this circuit? The solutions to the characteristic equation are complex so the response will be **underdamped**.

Ex. 2 Analysis of the circuit shown yields the characteristic equation below. The switch has been in the open position for a long time and is closed (as shown) at time $t = 0$. Find the initial and final conditions and write the full expression for $i_L(t)$, including all the constants that you find.



$$s^2 + \left(\frac{1}{C\cdot R_1}\right)\cdot s + \left(\frac{1}{L\cdot C}\right) = 0$$

$$\left(\frac{1}{C\cdot R_1}\right) = 1\cdot 10^4 \cdot \frac{1}{sec} \qquad \left(\frac{1}{L\cdot C}\right) = 2\cdot 10^7 \cdot \frac{1}{sec^2}$$

$$s^2 + 10000\cdot\frac{1}{sec}\cdot s + 2\cdot 10^7\cdot\frac{1}{sec^2} = 0$$

$$s_1 := \left[\frac{-10000}{2} + \frac{1}{2}\cdot\sqrt{(10000)^2 - 4\cdot\left(2\cdot 10^7\right)}\right]\cdot sec^{-1} \qquad s_2 := \left[\frac{-10000}{2} - \frac{1}{2}\cdot\sqrt{(10000)^2 - 4\cdot\left(2\cdot 10^7\right)}\right]\cdot sec^{-1}$$

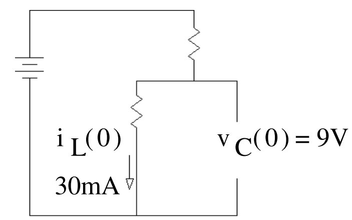
$$s_1 = -2764\cdot sec^{-1} \qquad s_2 = -7236\cdot sec^{-1}$$

s_1 and s_2 are both real and distinct, overdamped

Find the initial conditions:

Before the switch closed, the inductor current was: $\frac{15\cdot V}{R_1 + R_2} = 30\cdot mA = i_L(0)$

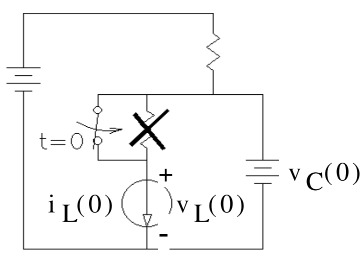
Before the switch closed, the capacitor voltage was: $\frac{R_2}{R_1 + R_2}\cdot(15\cdot V) = 9\cdot V = v_C(0)$



When the switch is closed, the inductor is suddenly in parallel with the capacitor, and:

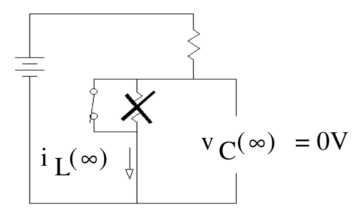
$$v_L(0) = v_C(0)$$

$$\frac{d}{dt}i_L(0) = \frac{1}{L}\cdot v_L(0) = \frac{1}{L}\cdot 9\cdot V = 90\cdot\frac{A}{sec}$$



Find the final condition:

$$i_L(\infty) = \frac{15\cdot V}{R_1} = 75\cdot mA$$



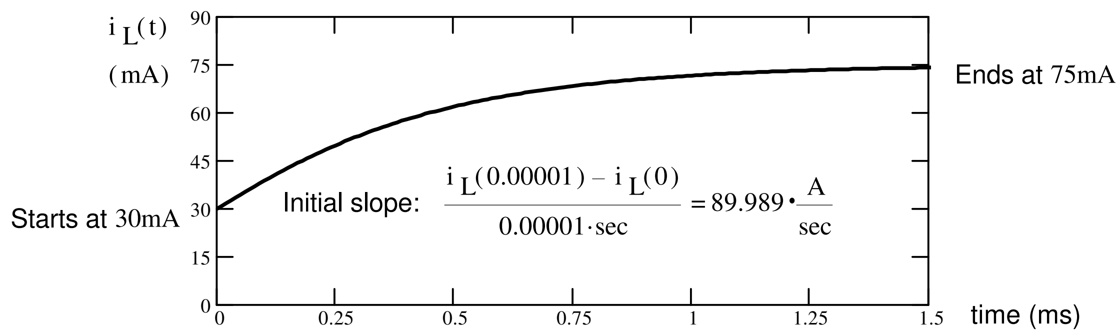
General solution for the overdamped condition: $i_L(t) = i_L(\infty) + B\cdot e^{s_1\cdot t} + D\cdot e^{s_2\cdot t}$

Initial conditions: $i_L(0) = \frac{15\cdot V}{R_1 + R_2} = i_L(\infty) + B + D$, so $B = i_L(0) - i_L(\infty) - D = 30\cdot mA - 75\cdot mA - D$
$= -45\cdot mA - D$

$$\frac{d}{dt}i_L(0) = 90\cdot\frac{A}{sec} = s_1\cdot B + s_2\cdot D = s_1\cdot(-45\cdot mA - D) + s_2\cdot D = s_1\cdot(-45\cdot mA) - s_1\cdot D + s_2\cdot D$$

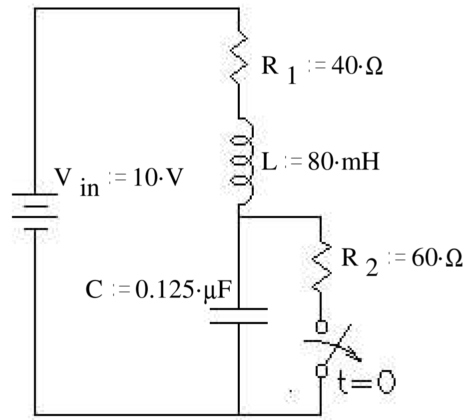
solve for D & B: $D := \frac{90\cdot\frac{A}{sec} - s_1\cdot(-45\cdot mA)}{-s_1 + s_2}$ $\qquad D = 7.69\cdot mA \qquad B := -45\cdot mA - D \qquad B = -52.7\cdot mA$

Plug numbers back in: $i_L(t) := 75\cdot mA - 52.7\cdot mA\cdot e^{-2764t} + 7.69\cdot mA\cdot e^{-7236t}$



Ex. 3

Analysis of the circuit shown yields the characteristic equation and s values below. The switch has been in the closed position for a long time and is opened (as shown) at time $t = 0$. Find the initial and final conditions and write the full expression for $v_C(t)$, including all the constants.



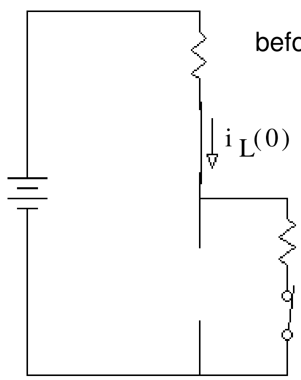
$$0 = s^2 + \frac{R_1}{L}\cdot s + \frac{1}{L\cdot C}$$

$$s_1 := \left(-250 + 10^4\cdot j\right)\cdot\frac{1}{sec} \quad , \qquad s_2 := \left(-250 - 10^4\cdot j\right)\cdot\frac{1}{sec}$$

Solution: $\alpha := -250\cdot\frac{1}{sec}$ $\qquad \omega := 10000\cdot\frac{rad}{sec}$

Initial conditions:

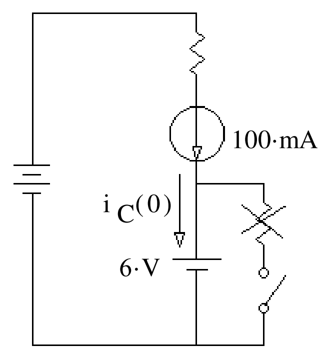
before switch opens



$$i_L(0) = \frac{V_{in}}{R_1 + R_2} = 100 \cdot mA$$

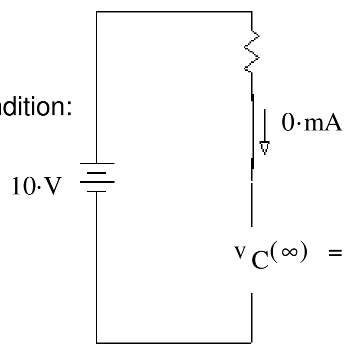
$$v_C(0) = V_{in}\cdot\frac{R_2}{R_1 + R_2} = 6 \cdot V$$

just after the switch opens



$$\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C} = \frac{100\cdot mA}{C} = 8\cdot 10^5 \cdot \frac{V}{sec}$$

Find final condition:



$$v_C(\infty) = V_{in} = 10 \cdot V$$

Find constants: $v_C(0) = v_C(\infty) + B$ $\qquad B = v_C(0) - v_C(\infty)$ $\qquad B := 6\cdot V - 10\cdot V$ $\qquad B = -4 \cdot V$

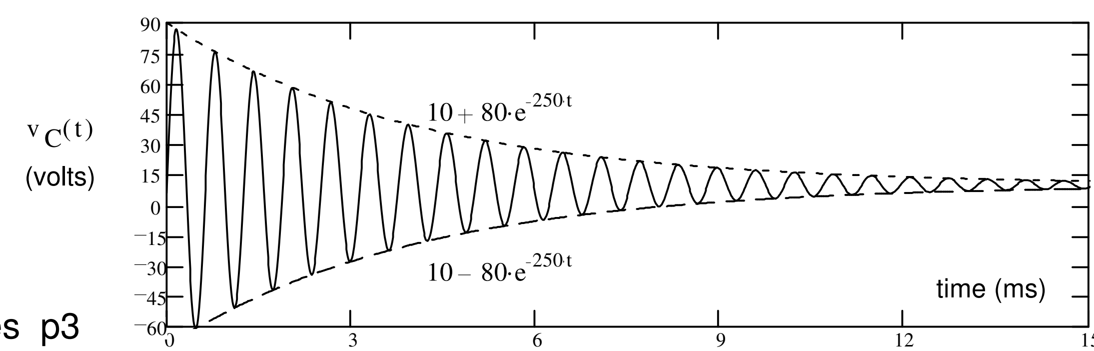
$$\frac{d}{dt}v_C(0) = \alpha\cdot B + D\cdot\omega \qquad D := \frac{8\cdot 10^5\cdot\frac{V}{sec} - \alpha\cdot B}{\omega} \qquad D = 79.9 \cdot V$$

Write the full expression for $v_C(t)$, including all the constants that you find.

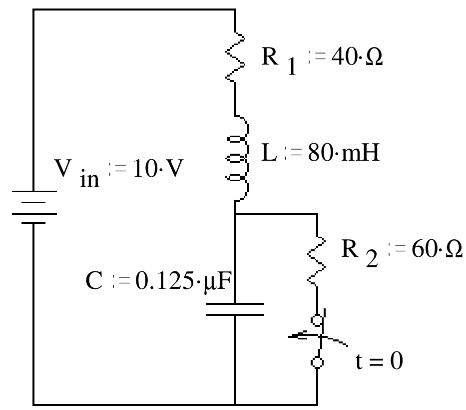
$$v_C(t) = e^{\alpha t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t)) + v_C(\infty)$$

$$v_C(t) := e^{-250t}\cdot\left(-4\cdot V\cdot\cos\left(10^4\cdot t\right) + 79.9\cdot V\cdot\sin\left(10^4\cdot t\right)\right) + 10\cdot V$$

$$\sqrt{D^2 + B^2} = 80 \cdot V$$



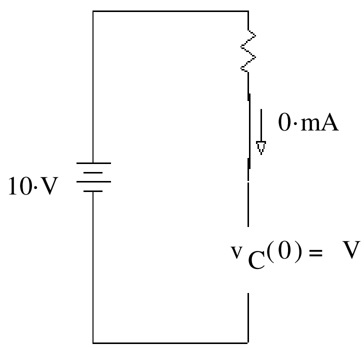
Ex. 4   Ex.3 Backwards, switch closes at t = 0



Characteristic eq.: $0 = s^2 + \left(\frac{1}{C \cdot R_2} + \frac{R_1}{L}\right) \cdot s + \left(1 + \frac{R_1}{R_2}\right) \cdot \frac{1}{L \cdot C}$

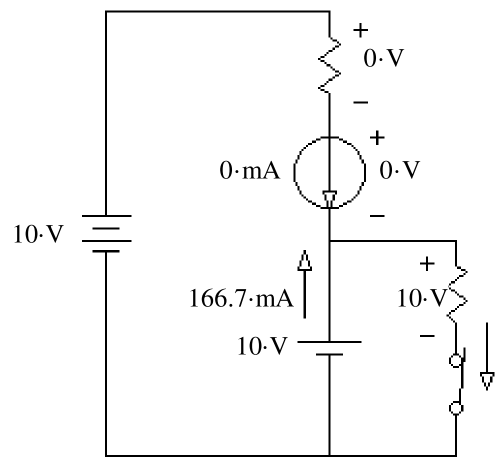
$s_1 := -1.257 \cdot 10^3 \cdot \frac{1}{sec}$ $s_2 := -1.326 \cdot 10^5 \cdot \frac{1}{sec}$

Initial conditions, same as Ex.3 final:



$v_C(0) = V_{in} = 10 \cdot V$

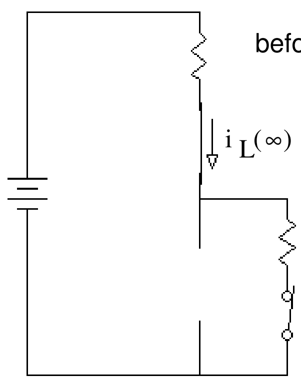
just after the switch opens



$\frac{10 \cdot V}{R_2} = 166.7 \cdot mA$

$\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C} = \frac{-166.7 \cdot mA}{C} = -1.334 \cdot 10^6 \cdot \frac{V}{sec}$

Find final condition:



before switch opens

$i_L(\infty) = \frac{V_{in}}{R_1 + R_2} = 100 \cdot mA$

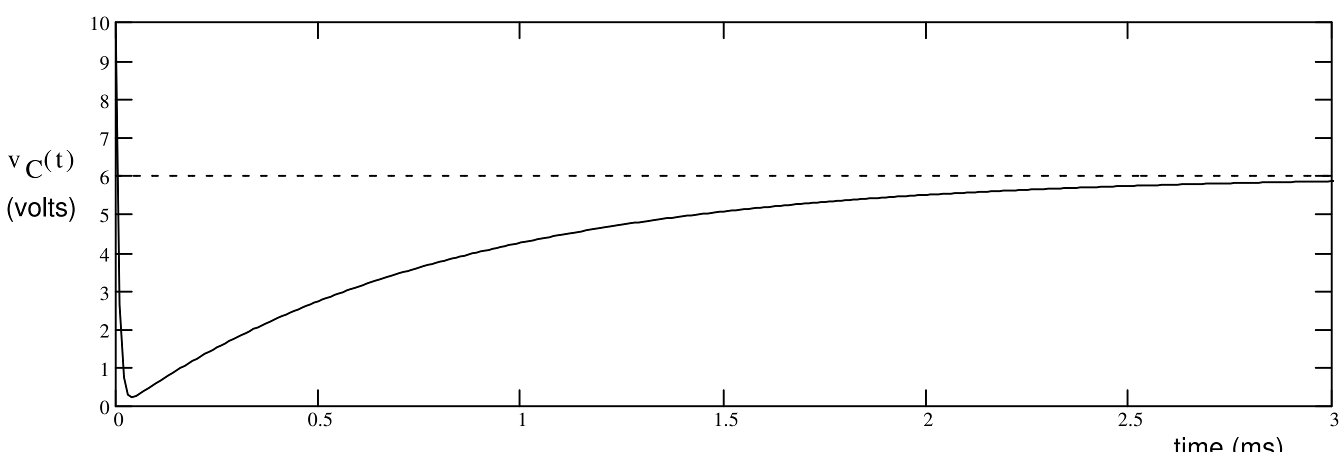
$v_C(\infty) = V_{in} \cdot \frac{R_2}{R_1 + R_2} = 6 \cdot V$

Find constants: $v_C(0) = v_C(\infty) + B + D$, so $B = v_C(0) - v_C(\infty) - D = 10 \cdot V - 6 \cdot V - D = 4 \cdot V - D$

$$\frac{d}{dt} v_C(0) = -1.334 \cdot 10^6 \cdot \frac{V}{sec} = s_1 \cdot B + s_2 \cdot D = s_1 \cdot (4 \cdot V - D) + s_2 \cdot D = s_1 \cdot (4 \cdot V) - s_1 \cdot D + s_2 \cdot D$$

$$D := \frac{-1.334 \cdot 10^6 \cdot \frac{V}{sec} - s_1 \cdot (4 \cdot V)}{-s_1 + s_2} \qquad D = 10.12 \cdot V \qquad B := 4 \cdot V - D \qquad B = -6.12 \cdot V$$

$$v_C(t) := 6 \cdot V - 6.12 \cdot V \cdot e^{-1257 t} + 10.12 \cdot V \cdot e^{-132600 \cdot t}$$



Ex. 5 Analysis of a circuit (not pictured) yields the characteristic equation below.

$$0 = s^2 + 400\cdot s + 400000 \qquad R := 80\cdot\Omega \qquad L := 20\cdot mH \qquad C := 2\cdot\mu F$$

Further analysis yields the followiing initial and final conditions:

$$i_L(0) = 120\cdot mA \qquad v_L(0) = -3\cdot V \qquad v_C(0) = 7\cdot V \qquad i_C(0) = -80\cdot mA$$

$$i_L(\infty) = 800\cdot mA \qquad v_L(\infty) = 0\cdot V \qquad v_C(\infty) = 12\cdot V \qquad i_C(\infty) = 0\cdot mA$$

Write the full expression for $i_L(t)$, including all the constants that you find. $\qquad i_L(t) = ?$

Solution:

$$\frac{400}{2} = 200 \qquad \frac{\sqrt{400^2 - 4\cdot 400000}}{2} = 600j$$

$$s_1 := (-200 + 600\cdot j)\cdot\frac{1}{sec} \quad \text{and} \quad s_2 := (-200 - 600\cdot j)\cdot\frac{1}{sec}$$

$$\alpha := \mathrm{Re}(s_1) \qquad \alpha = -200\cdot sec^{-1} \qquad \omega := \mathrm{Im}(s_1) \qquad \omega = 600\cdot sec^{-1}$$

Initial slope: $\qquad \frac{d}{dt}i_L(0) = \frac{v_L(0)}{L} = \frac{-3\cdot V}{L} = -150\cdot\frac{A}{sec}$

General solution for the underdamped condition: $i_L(t) = i_L(\infty) + e^{\alpha\cdot t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t))$

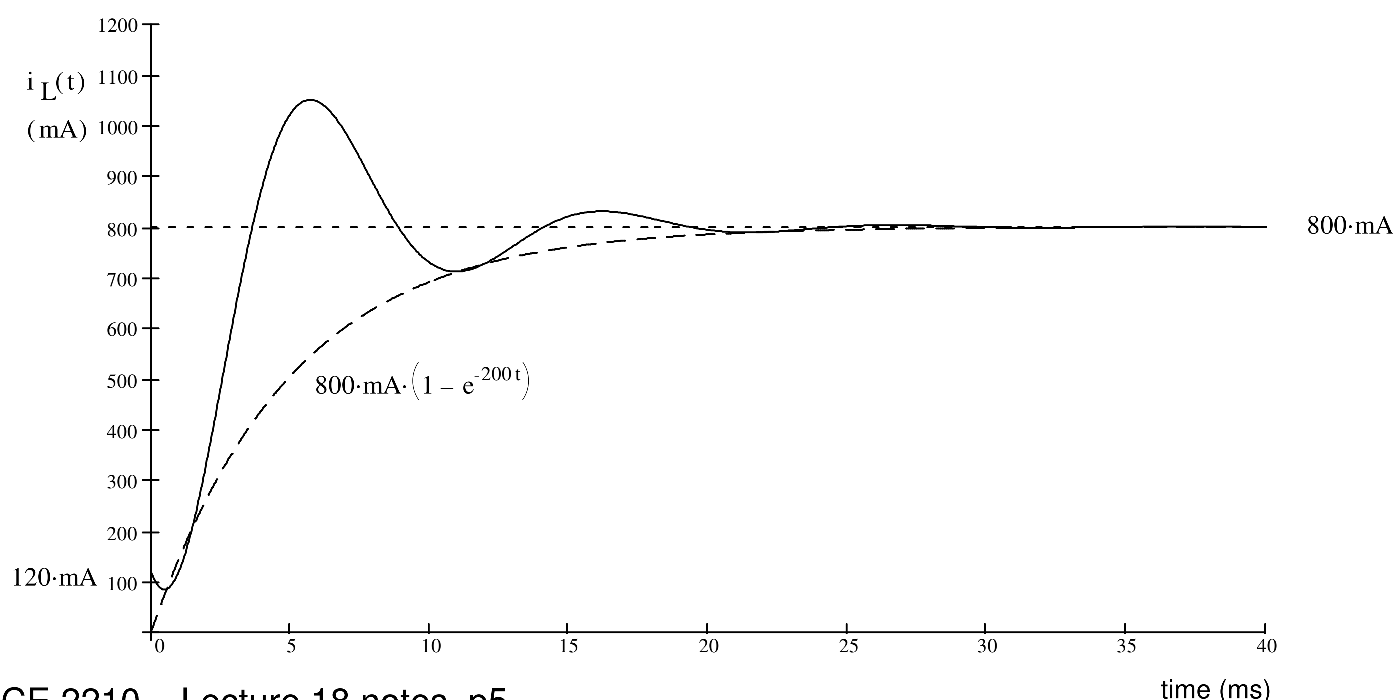
Find constants: $\qquad i_L(0) = i_L(\infty) + B \qquad B = i_L(0) - i_L(\infty) \qquad B := 120\cdot mA - 800\cdot mA$

$$B = -680\cdot mA$$

$$\frac{d}{dt}i_L(0) = \alpha\cdot B + D\cdot\omega \qquad D := \frac{-150\cdot\frac{A}{sec} - \alpha\cdot B}{\omega} \qquad D = -476.667\cdot mA$$

Write the full expression for $i_L(t)$, including all the constants that you find.

$$i_L(t) := 800\cdot mA + e^{-200t}\cdot(-680\cdot mA\cdot\cos(600\cdot t) - 477\cdot mA\cdot\sin(600\cdot t))$$



Ex. 6

Analysis of a circuit (not pictured) yields the characteristic equation below.

$0 = s^2 + 800\cdot s + 160000$ $\qquad R := 60\cdot\Omega \quad L := 350\cdot mH \quad C := 20\cdot\mu F \quad V_{in} := 12\cdot V$

Further analysis yields the following initial and final conditions:

$i_L(0) = 30\cdot mA \qquad v_L(0) = -7\cdot V \qquad v_C(0) = 5\cdot V \qquad i_C(0) = 70\cdot mA$

$i_L(\infty) = 90\cdot mA \qquad v_L(\infty) = 0\cdot V \qquad v_C(\infty) = 12\cdot V \qquad i_C(\infty) = 0\cdot mA$

Write the full expression for $i_L(t)$, including all the constants that you find. $\quad i_L(t) = ?$

Include **units** in your answer

Solution:

$\dfrac{-800 + \sqrt{800^2 - 4\cdot 160000}}{2} = -400 \qquad s_1 := -400\cdot\dfrac{1}{sec} \qquad s_2 := -400\cdot\dfrac{1}{sec}$

s_1 and s_2 are the same, **critically damped**

Initial slope: $\dfrac{d}{dt} i_L(0) = \dfrac{v_L(0)}{L} = \dfrac{-7\cdot V}{L} = -20\cdot\dfrac{A}{sec}$

General solution for the critically damped condition: $i_L(t) = i_L(\infty) + B\cdot e^{s_1\cdot t} + D\cdot t\cdot e^{s_2\cdot t}$

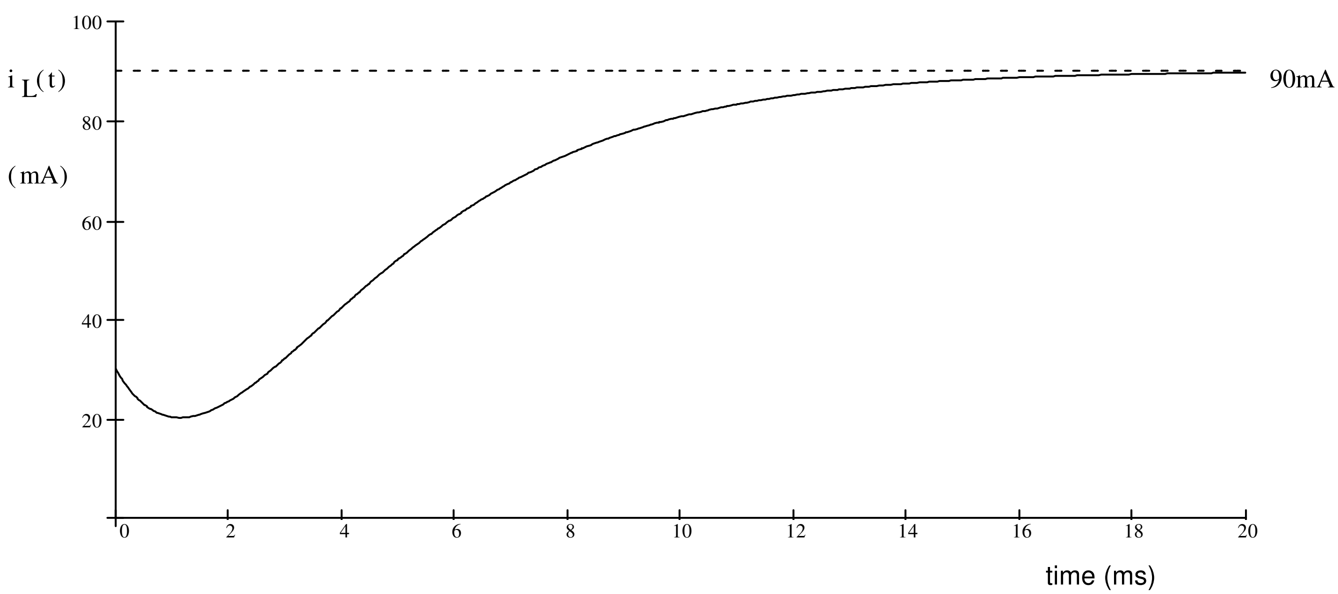
Find constants: $\quad i_L(0) = i_L(\infty) + B \qquad B = i_L(0) - i_L(\infty) \qquad B := 30\cdot mA - 90\cdot mA$

$B = -60\cdot mA$

$\dfrac{d}{dt} i_L(0) = B\cdot s + D \qquad D := -20\cdot\dfrac{A}{sec} - B\cdot s_1 \qquad D = -44\cdot\dfrac{A}{sec}$

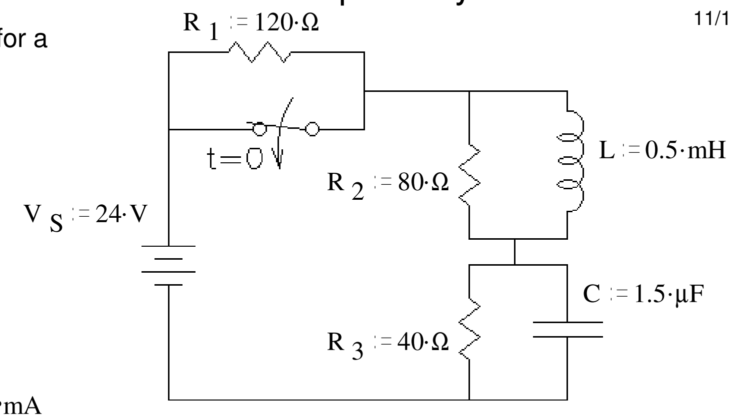
Write the full expression for $i_L(t)$, including all the constants that you find.

$$i_L(t) := 90\cdot mA - 60\cdot mA\cdot e^{-\frac{400}{sec}\cdot t} - 44\cdot\frac{A}{sec}\cdot t\cdot e^{-\frac{400}{sec}\cdot t}$$

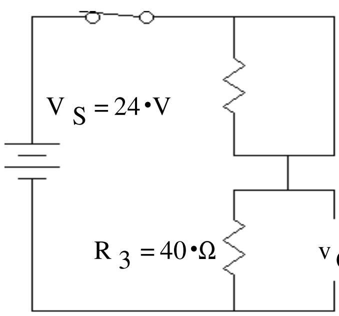


A. Stolp
11/1/06

Ex 1. The switch at right has been in the open position for a long time and is closed (as shown) at time $t = 0$.



a) What are the final conditions of i_L and the v_C?



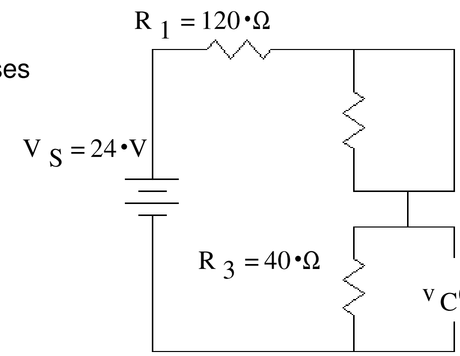
$R_3 := 40\cdot\Omega$

$$i_L(\infty) = \frac{V_S}{R_3} = 600\cdot mA$$

$$v_C(\infty) = V_S = 24\cdot V$$

b) Find the initial condition and intial slope of i_L so that you could find all the constants in $i_L(t)$.
Don't find $i_L(t)$ or it's constants, just the initial conditions.

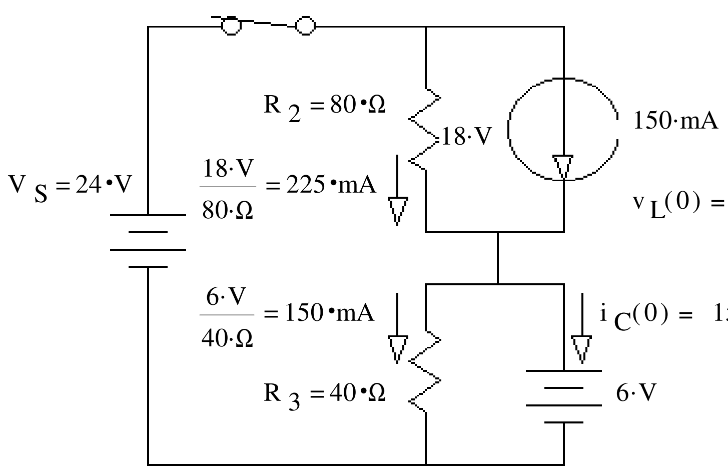
Before the switch closes



$$i_L(0) = \frac{V_S}{R_1 + R_3} = 150\cdot mA$$

$$v_C(0) = V_S\cdot\frac{R_3}{R_1 + R_3} = 6\cdot V$$

Just after the switch closes:



$$v_L(0) = 24\cdot V - 6\cdot V = 18\cdot V$$

$$\frac{d}{dt}i_L(0) = \frac{18\cdot V}{L} = 36000\cdot\frac{A}{sec}$$

$$i_C(0) = 150\cdot mA + \frac{18\cdot V}{R_2} - \frac{6\cdot V}{R_3} = 225\cdot mA$$

c) Find the initial condition and intial slope of v_C so that you could find all the constants in $v_C(t)$.
Don't find $v_C(t)$ or it's constants, just the initial conditions.

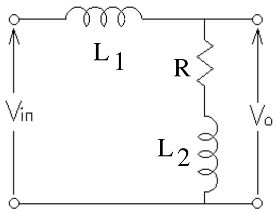
$$v_C(0) = V_S\cdot\frac{R_3}{R_1 + R_3} = 6\cdot V$$

$$\frac{d}{di}v_C(0) = \frac{225\cdot mA}{C} = 150000\cdot\frac{V}{sec}$$

Systems

Now that we' ve developed the concept of the transfer function, we can now develop system block diagrams using blocks which contain transfer functions.

Consider a circuit:



$$\mathbf{H}(s) = \frac{\mathbf{V}_{\mathbf{o}}(s)}{\mathbf{V}_{\mathbf{in}}(s)} = \frac{R + L_2 \cdot s}{R + L_1 \cdot s + L_2 \cdot s} = \frac{R + L_2 \cdot s}{R + (L_1 + L_2) \cdot s}$$

$$= \frac{L_2 \cdot s + R}{(L_1 + L_2) \cdot s + R}$$

This could be represented in as a block operator:

$$\mathbf{V}_{\mathbf{in}}(s) \longrightarrow \boxed{\frac{L_2 \cdot s + R}{(L_1 + L_2) \cdot s + R}} \longrightarrow \mathbf{V}_{\mathbf{o}}(s) = \mathbf{V}_{\mathbf{in}}(s) \cdot \mathbf{H}(s)$$

Transfer functions can be written for all kinds of devices and systems, not just electric circuits and the input and output do not have to be similar. For instance, the potentiometers used to measure angular position in the lab servo can be represented like this:

$$\boldsymbol{\theta}_{\mathbf{in}}(s) \longrightarrow \boxed{Kp = 0.7 \cdot \frac{V}{rad} = 0.012 \cdot \frac{V}{deg}} \longrightarrow \mathbf{V}_{\mathbf{out}}(s) = K_p \cdot \boldsymbol{\theta}_{\mathbf{in}}(s)$$

In general:

$$\mathbf{H}(s) = \frac{\mathbf{X}_{\mathbf{out}}(s)}{\mathbf{X}_{\mathbf{in}}(s)} \qquad \mathbf{X}_{\mathbf{in}}(s) \longrightarrow \boxed{\mathbf{H}(s)} \longrightarrow \mathbf{X}_{\mathbf{out}}(s) = \mathbf{X}_{\mathbf{in}}(s) \cdot \mathbf{H}(s)$$

$\mathbf{X}_{\mathbf{in}}$ and $\mathbf{X}_{\mathbf{out}}$ could be anything from small electrical signals to powerful mechanical motions or forces.

Two blocks with transfer functions $\mathbf{A}(s)$ and $\mathbf{B}(s)$ in a row would look like this:

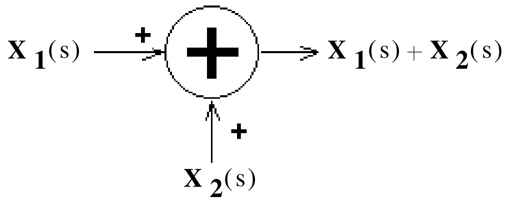
$$\underbrace{\mathbf{X}_{\mathbf{in}}(s) \longrightarrow \boxed{\mathbf{A}(s)} \xrightarrow{\mathbf{X}_{\mathbf{in}}(s) \cdot \mathbf{A}(s)} \boxed{\mathbf{B}(s)}}_{} \longrightarrow \mathbf{X}_{\mathbf{out}}(s) = \mathbf{X}_{\mathbf{in}}(s) \cdot \mathbf{A}(s) \cdot \mathbf{B}(s)$$

The two blocks could be replaced by a single equivalent block:

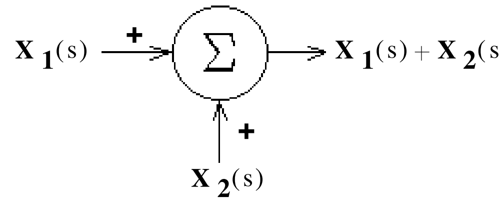
$$\mathbf{X}_{\mathbf{in}}(s) \longrightarrow \boxed{\mathbf{A}(s) \cdot \mathbf{B}(s)} \longrightarrow \mathbf{X}_{\mathbf{out}}(s) = \mathbf{X}_{\mathbf{in}}(s) \cdot \mathbf{A}(s) \cdot \mathbf{B}(s)$$

$$\mathbf{X}_{\mathbf{in}}(s) \cdot \mathbf{A}(s)$$

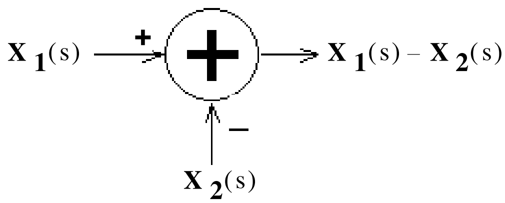
Summer blocks can be used to add signals:



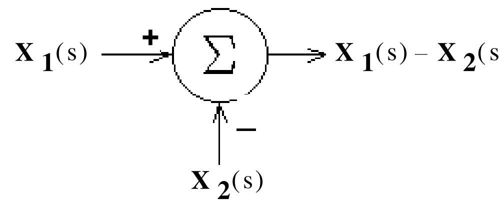
OR



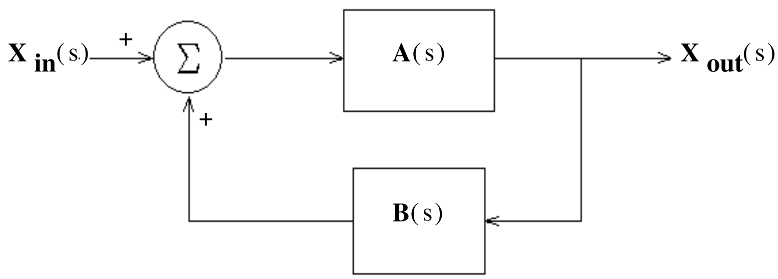
or subtract signals:



OR

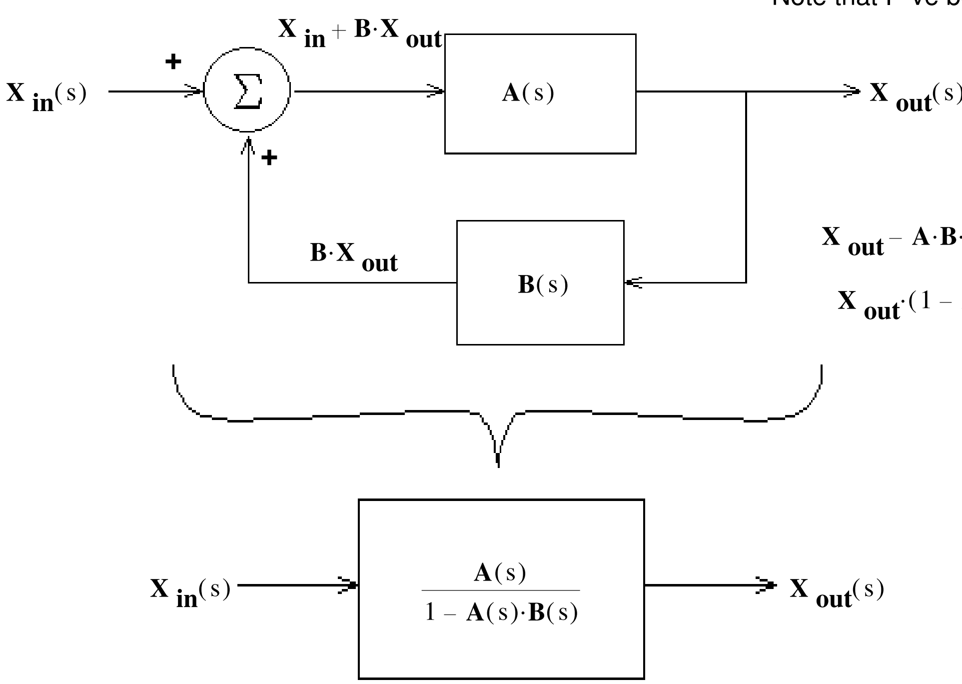


A feedback loop system is particularly interesting and useful:



The entire loop can be replaced by a single equivalent block:

Note that I' ve begun to drop the (s)



$$X_{out}(s) = A \cdot \left(X_{in} + B \cdot X_{out}\right)$$

$$= A \cdot X_{in} + A \cdot B \cdot X_{out}$$

$$X_{out} - A \cdot B \cdot X_{out} = A \cdot X_{in}$$

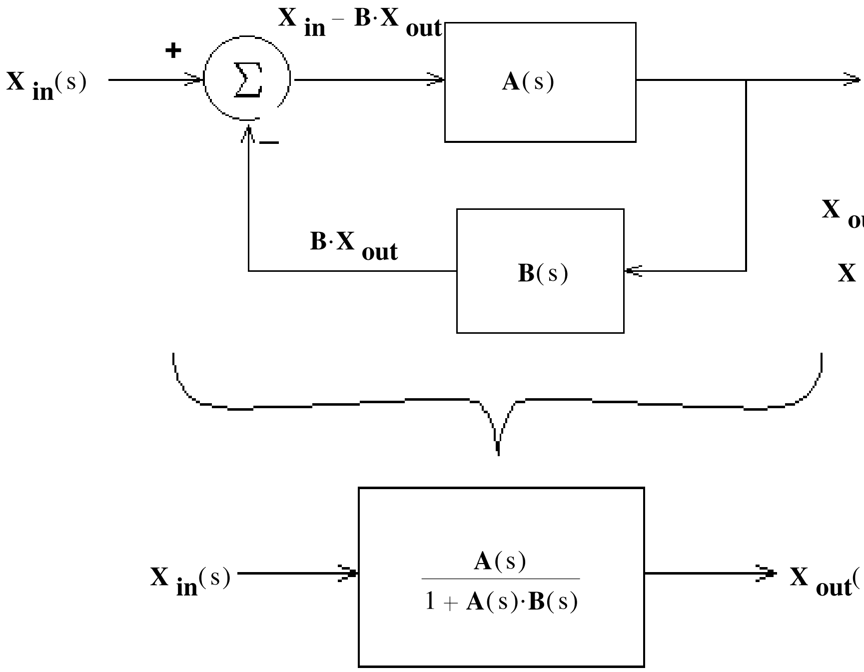
$$X_{out} \cdot (1 - A \cdot B) = A \cdot X_{in}$$

$$\frac{X_{out}}{X_{in}} = \frac{A}{1 - A \cdot B} = H(s)$$

The equivalent transfer function

$A(s) \cdot B(s)$ is called the "loop gain" or "open loop gain"

Negative feedback is more common and is used as a control system:



$$X_{out}(s) = A\cdot\left(X_{in} - B\cdot X_{out}\right)$$

$$= A\cdot X_{in} - A\cdot B\cdot X_{out}$$

$$X_{out} + A\cdot B\cdot X_{out} = A\cdot X_{in}$$

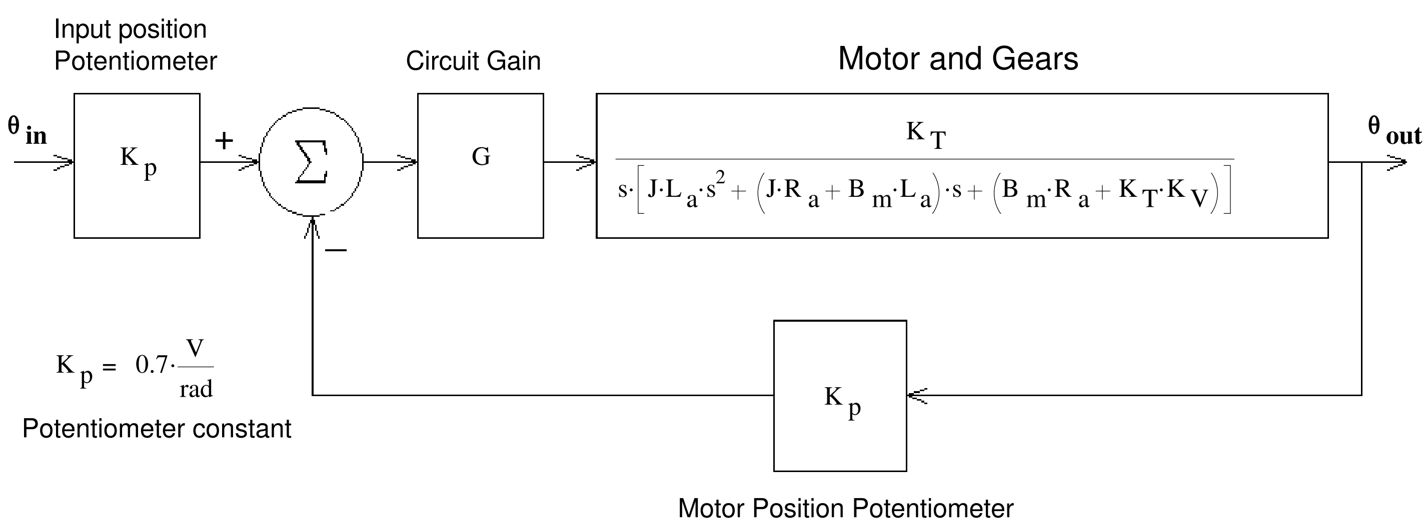
$$X_{out}\cdot(1 + A\cdot B) = A\cdot X_{in}$$

$$\frac{X_{out}}{X_{in}} = \frac{A}{1 + A\cdot B} = H(s)$$

The equivalent transfer function

This is called a "closed loop" system, whereas a a system without feedback is called "open loop". The term "open loop" is often used to describe a system that is out of control.

The servo used in our lab can be represented by:



$$H(s) = \frac{\theta_{out}(s)}{\theta_{in}(s)} = \frac{G\cdot K_T\cdot K_p}{s\cdot\left[J\cdot L_a\cdot s^2 + \left(J\cdot R_a + B_m\cdot L_a\right)\cdot s + \left(B_m\cdot R_a + K_T\cdot K_V\right)\right] + K_p\cdot G\cdot K_T}$$