Recall Statics. To find the reaction at each support, s to each load on a beam (or anything else) can be found separately for each load. the total reactions are simply the sum of the

\[ P_1 + P_2 \]

Superposition

For circuits with more than 1 source.

1) Zero all but one source. (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
2) Compute your wanted voltage or current due to the remaining source. Careful, some may be negative.
3) Repeat the first two steps for all the sources.
4) Sum all the contributions from all the sources to find the actual voltage or current. Watch your signs!

**Ex1.** Use the method of superposition to find the current \( I_2 \) (through \( R_2 \)) and the voltage across \( R_1 \) \( (V_{R1}) \). Be sure to clearly show and circle your intermediate results.

superposition:

Eliminate current source

\[ I_{2,VS} := \frac{V_S}{R_1 + R_2} \quad I_{2,VS} = 20\text{mA} \]

\[ V_{R1,VS} := \frac{R_1}{R_1 + R_2} \cdot V_S \quad V_{R1,VS} = 2\text{V} \]

Eliminate voltage source

\[ I_{2,IS} := \frac{1}{R_2} \cdot I_S \quad I_{2,IS} = 6\text{mA} \]

\[ V_{R1,IS} := -I_{2,IS} \cdot R_2 \quad V_{R1,IS} = 1.2\text{V} \]

Add results

\[ I_2 = I_{2,VS} + I_{2,IS} \quad I_2 = 14\text{mA} \]

\[ V_{R1} = V_{R1,VS} + V_{R1,IS} \quad V_{R1} = 3.2\text{V} \]
Ex2. Use the method of superposition to find the voltage across through $R_2$ and the current through $R_3$. Be sure to clearly show and circle your intermediate results.

Eliminate current source
$R_1$ is a separate path and doesn't matter.

$$V_{R2.VS} = \frac{R_2}{R_2 + R_3} \cdot V_S$$

$$V_{R2.VS} = 4.8 \cdot V$$

$$I_{R3.VS} = \frac{V_S}{R_2 + R_3}$$

$$I_{R3.VS} = -2.4 \cdot mA$$

Eliminate voltage source
$R_1$ is shorted and doesn't matter.

$$V_{R2.Is} := I_S \frac{1}{\frac{1}{R_2} + \frac{1}{R_3}}$$

$$V_{R2.Is} = 2.4 \cdot V$$

$$I_{R3.Is} := I_S \frac{1}{\frac{1}{R_3} + \frac{1}{R_2}}$$

$$I_{R3.Is} = 0.8 \cdot mA$$

Add results

$$V_{R2} := V_{R2.VS} + V_{R2.Is}$$

$$V_{R2} = 7.2 \cdot V$$

$$I_{R3} := I_{R3.VS} + I_{R3.Is}$$

$$I_{R3} = -1.6 \cdot mA$$
Model of a Real Source

Real sources are not ideal, but we will model them with two ideal components.

![Diagram of a real source model](image)

- \( V_{\text{term}} \) is the variable voltage source.
- \( I_{\text{term}} \) is the variable current source.
- \( R_S \) is the series resistor.
- \( R_L \) is the load resistor.
- \( V_L \) is the voltage across the load.
- \( I_L \) is the current through the load.

Note: \( R_L \) is NOT part of the Thévenin equivalent circuit and does not need to be shown.

Thévenin Equivalent Circuit

The same model can be used for any combination of sources and resistors.

![Diagram of a Thévenin equivalent circuit](image)

Thévenin Equivalent:

To calculate a circuit's Thévenin equivalent:

1) Remove the load and calculate the open-circuit voltage where the load used to be. This is the Thévenin voltage (\( V_{\text{Th}} \)).

2) Zero all the sources.
   (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)

3) Compute the total resistance between the load terminals.
   (DO NOT include the load in this resistance.) This is the Thévenin source resistance (\( R_{\text{Th}} \)).

4) Draw the Thévenin equivalent circuit and add your values.
Norton equivalent

To calculate a circuit's Norton equivalent:

1) Replace the load with a short (a wire) and calculate the short-circuit current in this wire.
   This is the Norton current ($I_N$). Remove the short.

2) Zero all the sources.
   (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)

3) Compute the total resistance between the load terminals.
   (DO NOT include the load in this resistance.) This is the Norton source resistance ($R_N$).
   (Exactly the same as the Thévenin source resistance ($R_{Th}$)).

4) Draw the Norton equivalent circuit and add your values.

OR (the more common way)...

1) Find the Thévenin equivalent circuit.

2) Convert to Norton circuit, then >>>

   $R_N = R_{Th}$  \hspace{1cm}  and \hspace{1cm}  $I_N = \frac{V_{Th}}{R_{Th}}$
Ex 1 Find the Thévenin equivalent:

\[
V_S := 20 \text{ V}
\]

\[
R_1 := 40 \, \Omega
\]

\[
R_2 := 120 \, \Omega
\]

\[
R_L := 60 \, \Omega
\]

To calculate a circuit’s Thévenin equivalent:
1) Remove the load and calculate the open-circuit voltage where the load used to be. This is the Thévenin voltage \(V_{Th}\).

\[
V_{oc} = V_{Th} = V_S \frac{R_2}{R_1 + R_2}
\]

\[
V_{Th} = 15 \, \text{V}
\]

2) Zero all the sources. (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)

Find the open circuit voltage:

3) Compute the total resistance between the load terminals. (DO NOT include the load in this resistance.) This is the Thévenin source resistance \(R_{Th}\).

\[
R_{Th} := \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}
\]

\[
R_{Th} = 30 \, \Omega
\]

4) Draw the Thévenin equivalent circuit and add your values.

Thevenin equivalent circuit:

\[
V_{Th} = 15 \, \text{V}
\]

If the load were reconnected:

\[
V_{Th} = 15 \, \text{V}
\]

\[
R_L = 60 \, \Omega
\]

\[
V_L = V_{Th} \frac{R_L}{R_{Th} + R_L} = 10 \, \text{V}
\]

\[
I_L = \frac{V_{Th}}{R_{Th} + R_L} = 166.7 \, \text{mA}
\]

\[
P_L = 10 \cdot \text{V} \cdot 166.7 \, \text{mA} = 1667 \, \text{W}
\]

b) Find the Norton equivalent circuit:

\[
I_N := \frac{V_S}{R_1}
\]

\[
I_N = 500 \, \text{mA}
\]

\[
R_1 := 40 \, \Omega
\]

\[
S = 20 \, \text{V}
\]

Norton equivalent circuit:

\[
I_N := \frac{V_{Th}}{R_{Th}}
\]

\[
I_N = 500 \, \text{mA}
\]

\[
R_N := R_{Th}
\]

\[
R_N = 30 \, \Omega
\]
c) Show that the Thévenin circuit is indeed equivalent to the original at several values of $R_L$.

<table>
<thead>
<tr>
<th>$R_L$</th>
<th>$V_{I,\text{th}}$</th>
<th>$I_{I,\text{th}}$</th>
<th>$V_L$</th>
<th>$I_L$</th>
<th>$P_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \Omega$</td>
<td>500 mA</td>
<td>500 mA</td>
<td>0 V</td>
<td>500 mA</td>
<td>0 W</td>
</tr>
<tr>
<td>$10 \Omega$</td>
<td>$\frac{1}{R_2 + \frac{1}{R_L}}$</td>
<td>$\frac{V_S}{R_1} = 3.75 V$</td>
<td>$I_L = \frac{V_L}{R_L} = 375 mA$</td>
<td>$P_L = V_L \cdot I_L = 1.406 W$</td>
<td></td>
</tr>
</tbody>
</table>

Repeat these calculations for a number of load resistors:

<table>
<thead>
<tr>
<th>$R_L$</th>
<th>$R_{o,\text{th}}$</th>
<th>$V_{I,\text{th}}$</th>
<th>$I_{I,\text{th}}$</th>
<th>$V_L$</th>
<th>$I_L$</th>
<th>$P_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \Omega$</td>
<td>0.992</td>
<td>0</td>
<td>483.871</td>
<td>375</td>
<td>6</td>
<td>1.8</td>
</tr>
<tr>
<td>$1 \Omega$</td>
<td>9.231</td>
<td>3.75</td>
<td>375</td>
<td>300</td>
<td>6</td>
<td>1.8</td>
</tr>
<tr>
<td>$10 \Omega$</td>
<td>17.143</td>
<td>6</td>
<td>300</td>
<td>250</td>
<td>7.5</td>
<td>1.875</td>
</tr>
<tr>
<td>$20 \Omega$</td>
<td>24</td>
<td>7.5</td>
<td>250</td>
<td>100</td>
<td>12</td>
<td>1.2</td>
</tr>
<tr>
<td>$30 \Omega$</td>
<td>30</td>
<td>8.571</td>
<td>214.286</td>
<td>166.667</td>
<td>10</td>
<td>1.667</td>
</tr>
<tr>
<td>$40 \Omega$</td>
<td>40</td>
<td>10</td>
<td>166.667</td>
<td>166.667</td>
<td>10</td>
<td>1.667</td>
</tr>
<tr>
<td>$60 \Omega$</td>
<td>60</td>
<td>12</td>
<td>100</td>
<td>100</td>
<td>12</td>
<td>1.2</td>
</tr>
<tr>
<td>$120 \Omega$</td>
<td>80</td>
<td>13.333</td>
<td>55.556</td>
<td>55.556</td>
<td>13.333</td>
<td>0.741</td>
</tr>
<tr>
<td>$240 \Omega$</td>
<td>120</td>
<td>15</td>
<td>55.556</td>
<td>0</td>
<td>15</td>
<td>0</td>
</tr>
</tbody>
</table>

Plots:

- Power delivered to the load ($R_L$) as a function of $R_L$.
Maximum power transfer

If I wanted to maximize the power dissipated by the load, what $R_L$ would I choose?

Maximum power transfer happens when: $R_L = R_S$

Just what we saw in Example 1

This is rarely important in power circuitry, where there should be plenty of power and $R_S$ should be small. It is much more likely to be important in signal circuitry where the voltages can be very small and the source resistance may be significant -- say a microphone or a radio antenna.

All you need to remember is: $R_L = R_S$ to maximize the power dissipation in $R_L$

What about efficiency?

The bigger $R_L$ is, the higher the efficiency.
Ex 2  a) Find and draw the Thévenin equivalent circuit.

First do some simplification:

\[ R_{eq} = \frac{1}{\frac{1}{R_3} + \frac{1}{R_2 + R_4}} \]

Divide this voltage between \( R_2 \) and \( R_4 \):

\[ V_{234} = 9 \cdot V \quad V_{Th} = \frac{R_4}{R_2 + R_4} \cdot V_{234} \]

Find the Thévenin resistance:

\[ R_{Th} = 750 \cdot \Omega \]

Thévenin equivalent circuit:

\[ V_{Th} = 3 \cdot V \]

If the load were reconnected:

\[ V_L = V_{Th} \frac{R_L}{R_{Th} + R_L} \]

\[ I_L = \frac{V_{Th}}{R_{Th} + R_L} \]

\[ I_L = 2.5 \cdot mA \]

b) Find and draw the Norton equivalent circuit.

\[ I_N = \frac{V_{Th}}{R_{Th}} \]

\[ I_N = 4 \cdot mA \]

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c) Use your Norton equivalent circuit to find the current through the load.

\[ I_N = 4 \text{ mA}, \quad R_N = \frac{1}{I_N} = \frac{1}{2.5 \text{ mA}}, \quad R_L = 450 \Omega, \quad I_L = 1 \frac{V}{\Omega} \frac{mA}{mA} = 1.125 \Omega \]

\[ V_L := I_L \cdot R_L \quad V_L = 1.125 \cdot V \]

For maximum power transfer, \( R_L = R_{Th} = 750 \Omega \)

\[ I_L = \frac{V_{Th}}{2} \]

\[ P_L = \frac{V_L^2}{R_L} = 3 \text{ mW} \]

d) What value of \( R_L \) would result in the maximum power delivery to \( R_L \)?

\[ R_L = R_{Th} = 750 \Omega \]

e) What is the maximum power transfer?

\[ \text{Ex 3} \quad \text{a) Find and draw the Thévenin & Norton equivalent circuits.} \]

\[ V_{S1} = 10 \cdot V, \quad R_1 = 5 \Omega, \quad R_2 = 15 \Omega, \quad V_{S2} = 20 \cdot V, \quad R_L = 20 \Omega \]

\[ I = 0.5 \text{ A}, \quad V_{Th} = 10 \cdot V + 2.5 \cdot V \]

\[ V_{S1} = 10 \cdot V, \quad R_1 = 5 \Omega, \quad R_2 = 15 \Omega, \quad V_{S2} = 20 \cdot V \]

\[ V_{Th} = 12.5 \cdot V \]

\[ \text{Thévenin equivalent circuit:} \quad R_{Th} = 3.75 \cdot \Omega \]

\[ \text{Norton equivalent circuit:} \quad I_N = \frac{V_{Th}}{R_{Th}} = \frac{12.5}{3.75} \text{ A} = 3.333 \text{ A}, \quad R_N = 3.8 \cdot \Omega \]

\[ V_{Th} = 12.5 \cdot V, \quad R_L = 20 \Omega, \quad V_L = \frac{R_L}{R_{Th} + R_L} \cdot V_{Th} = 10.526 \cdot V \]

b) Use your Thévenin equivalent circuit to find the voltage across the load.
Ex 4  a) Find and draw the Thévenin & Norton equivalent circuits.

\[ V_S := 9 \cdot V \]

\[ R_1 := 40 \cdot \Omega \quad R_2 := 120 \cdot \Omega \]

\[ R_L := 72 \cdot \Omega \quad R_3 := 240 \cdot \Omega \]

\[ I_S := 50 \cdot mA \]

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Use superposition to find \( V_{Th} \).

\[ V_S = 9 \cdot V \]

\[ V_{Th.V} := \frac{R_2 + R_3}{R_1 + R_2 + R_3} \cdot V_S \]

\[ V_{Th.V} = 8.1 \cdot V \]

\[ V_{Th.I} := -I_{12} R_1 \quad V_{Th.I} = -1.2 \cdot V \]

\[ V_{Th} := V_{Th.V} + V_{Th.I} \quad V_{Th} = 6.9 \cdot V \]

Current divider:

\[ I_{12} = \frac{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \cdot I_S \]

\[ I_{12} = 30 \cdot mA \]

Find the Thévenin resistance

\[ R_1 = 40 \cdot \Omega \quad R_2 = 120 \cdot \Omega \]

\[ R_3 = 240 \cdot \Omega \]

\[ R_{Th} := \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \]

\[ R_{Th} = 36 \cdot \Omega \]

\[ I_L := \frac{V_{Th}}{R_{Th} + R_L} \]

\[ I_L = 63.889 \cdot mA \]

\[ V_L = I_L \cdot R_L = 4.6 \cdot V \]

Thévenin equivalent circuit:

\[ V_{Th} = 6.9 \cdot V \]

\[ R_{Th} = 36 \cdot \Omega \]

\[ \frac{V_{Th}}{R_{Th}} \]

Norton equivalent circuit:

\[ I_N := \frac{V_{Th}}{R_{Th}} \]

\[ I_N = 191.7 \cdot mA \]

\[ R_{N} := R_{Th} \]

\[ R_{N} = 36 \cdot \Omega \]
Ex 5 A NiCad Battery pack is used to power a cell phone. When the phone is switched on the battery pack voltage drops from $4.80\,\text{V}$ to $4.65\,\text{V}$ and the cell phone draws $50\,\text{mA}$.

(a) Draw a simple, reasonable model of the battery pack using ideal parts. Find the value of each part.

\[ R_S := \frac{V_S - V_{50}}{I_{50}} = 50\,\Omega \]

\[ V_S = 4.8\,\text{V} \]

\[ V_{50} = 4.65\,\text{V} \]

(b) The cell phone is used to make a call. Now it draws $300\,\text{mA}$. What is the battery pack voltage now?

\[ V_B = V_S - I_{\text{call}} R_S = 4.8\,\text{V} - 4.8\,\text{V} = 3.9\,\text{V} \]

(c) The battery pack is placed in a charger. The charger supplies $5.10\,\text{V}$. How much current flows into the battery pack?

\[ I_{\text{chg}} = \frac{V_{\text{chg}} - V_S}{R_S} = 100\,\text{mA} \]

Ex 6 Consider the circuit at right.

(a) What value of load resistor ($R_L$) would you choose if you wanted to maximize the power dissipation in that load resistor.

\[ R_L := R_S \quad R_L = 8\,\Omega \]

(b) With that load resistor ($R_L$) find the power dissipation in the load.

\[ I_L := \frac{I_S}{2} \quad P_L = I_L^2 R_L = 2\,\text{W} \]
Ex 7

Use superposition to find $V_{th}$.

$$V_S := 9\, \text{V}$$

$$I_S := 180\, \text{mA}$$

$$V_S := 9\, \text{V}$$

$$R_1 := 40\, \Omega$$

$$R_2 := 12.5\, \Omega$$

$$R_3 := 10\, \Omega$$

$$R_4 := 90\, \Omega$$

$$R_6 := 60\, \Omega$$

$$R_L := 30\, \Omega$$

Current divider:

$$I_{R4} := \frac{1}{R_2 + R_4 + R_6} I_S$$

$$I_{R4} = 35.556\, \text{mA}$$

$$V_{thI} := I_{R4} R_4$$

$$V_{thI} = 3.2\, \text{V}$$

Eliminate voltage source:

$$R_{246} := R_2 + R_4 + R_6$$

$$R_{246} = 162.5\, \Omega$$

$$V_{thV} := \frac{R_4}{R_1 + R_2 + R_4 + R_6} V_S$$

$$V_{thV} = 4\, \text{V}$$

Eliminate current source:

$$R_{th} := \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_6}} + R_3$$

$$R_{th} = 60\, \Omega$$

$$V_{th} := V_{thV} + V_{thI}$$

$$V_{th} = 7.2\, \text{V}$$

$${V_{th}}^2 = V_{thI}^2 + V_{thV}^2$$

$$I_{L} := \frac{V_{th}}{R_{th} + R_L}$$

$$I_{L} = 80\, \text{mA}$$

$$R_L = 30\, \Omega$$

$$V_L = I_L R_L = 2.4\, \text{V}$$

Norton equivalent circuit:

$$I_N := \frac{V_{th}}{R_{th}}$$

$$I_N = 120\, \text{mA}$$

$$R_N := R_{th}$$

$$R_N = 60\, \Omega$$

Thévenin equivalent circuit:

$$V_{Th} = 7.2\, \text{V}$$

$$R_{Th} = 60\, \Omega$$

Put the load back on:

$$R_{th} = 60\, \Omega$$

$$V_{Th} = 7.2\, \text{V}$$

$$I_L := \frac{V_{Th}}{R_{Th} + R_L}$$

$$I_L = 80\, \text{mA}$$

$$R_L = 30\, \Omega$$

$$V_L = I_L R_L = 2.4\, \text{V}$$
General Network Analysis

In many cases you have multiple unknowns in a circuit, say the voltages across multiple resistors. Network analysis is a systematic way to generate multiple equations which can be solved to find the multiple unknowns. These equations are based on basic Kirchoff’s and Ohm’s laws.

Loop or Mesh Analysis You may have used these methods in previous classes, particularly in Physics. The best thing to do now is to forget all that. Loop analysis is rarely the easiest way to analyze a circuit and is inherently confusing. Hopefully I’ve brought you to a stage where you have some intuitive feeling for how currents flow in circuits. I don’t want to ruin that now by screwing around with loop currents that don’t really exist.

Nodal analysis This is a much better method. It’s just as powerful, usually easier, and helps you develop your intuitive feeling for how circuits work.

Nodal Analysis

Node = all points connected by wire, all at same voltage (potential)

Ground: One node in the circuit which will be our reference node. Ground, by definition, will be the zero voltage node. All other node voltages will be referenced to ground and may be positive or negative. Think of gage pressure in a fluid system. In that case atmospheric pressure is considered zero. If there is no ground in the circuit, define one for yourself. Try to chose a node which is hooked to one side of a voltage source.

Nodal Voltage: The voltage of a node referenced to ground. The objective of nodal analysis is to find all the nodal voltages. If you know the voltage at a node then it’s a “known” node. Ground is a known node (duh, it’s zero). If one end of a known voltage source hooked to ground, then the node on the other end is also known (another duh).

Method: Label all the unknown nodes as; “a”, “b”, “c”, etc. Then the unknown nodal voltages become; \( V_a, V_b, V_c \), etc. Write a KCL equation for each unknown node, defining currents as necessary. Replace each unknown current with an Ohm’s law relationship using the nodal voltages. Now you have just as many equations as unknowns. Solve.

Nodal Analysis Steps

1) If the circuit doesn’t already have a ground, label one node as ground (zero voltage).
   If the ground can be defined as one side of a voltage source, that will make the following steps easier.
   Label the remaining nodes, either with known voltages or with letters, a, b, ...
2) Label unknown node voltages as \( V_a, V_b \), ... and label the current in each resistor as \( I_1, I_2, ... \)
3) Write Kirchhoff’s current equations for each unknown node.
4) Replace the currents in your KCL equations with expressions like this. \( \frac{V_a - V_b}{R_1} \) Ohm’s law relationship using the nodal voltages.
5) Solve the multiple equations for the multiple unknown voltages.

Nodal Analysis Examples

Ex 1 Use nodal analysis to find the voltage across \( R_1 \) \( (V_{R1}) \).

1) See next page
   Label one node as ground (zero voltage). By choosing the negative side of a voltage source as ground, the upper-left node is known \( (10\, \text{V}) \). Label the remaining nodes, either with known voltages or with letters, a, b, ...
2) Label unknown node voltages as \( V_a, V_b, \ldots \) and label the current in each resistor as \( I_1, I_2, \ldots \).

3) Write Kirchhoff's current equations for node a.
\[
I_1 + I_S = I_{R3}
\]

4) Replace the currents in the KCL equations with Ohm's law relationships.
\[
\frac{V_S - V_a}{R_1} + I_S = \frac{V_a - 0}{R_3}
\]
\[
\frac{V_S - V_a}{R_1} + I_S = \frac{V_a}{R_3}
\]

5) Solve:
\[
\frac{V_S}{R_1} + I_S = \frac{V_a}{R_3} + \frac{V_a}{R_1}
\]
\[
\frac{V_S}{R_1} + I_S = V_a \cdot \left( \frac{1}{R_1} + \frac{1}{R_3} \right)
\]
\[
\frac{V_S}{R_1} + I_S = V_a = 10.5 \text{ V}
\]
\[
V_{R1} := V_S - V_a \quad V_{R1} = -0.5 \text{ V}
\]

b) Find the current through \( R_3 \) (\( I_{R3} \)).
\[
I_{R3} = \frac{V_a}{R_3} = 3.5 \text{ mA}
\]

Ex 2 Same circuit used in Thévenin example, where \( R_4 \) was \( R_L \).

1) Define ground and nodes:
\[
\begin{align*}
V_S := 9 \text{ V} & \quad R_1 := 40 \Omega \\
R_2 := 120 \Omega & \quad R_4 := 72 \Omega \\
R_3 := 240 \Omega & \quad I_S := 50 \text{ mA}
\end{align*}
\]

2 unknown nodes --- will need 2 equations
2) Label unknown node voltages as $V_a$, $V_b$, ... and label the current in each resistor as $I_1$, $I_2$, .... It doesn't matter if these currents are in the correct directions.

3) Write Kirchoff's current equations for each unknown node.

node a: $I_1 = I_2 + I_4$
node b: $I_2 = I_3 + I_S$

Now you have just as many equations as unknowns.

4) Replace the currents in your KCL equations with expressions like this.

node a: $I_1 = \frac{V_a - V_b}{R_1}$
node b: $I_2 = \frac{V_a - V_b}{R_2} + \frac{V_a - 0 \cdot V}{R_4}$

Now you have just as many equations as unknowns.

5) Solve the multiple equations for the multiple unknown voltages. Solve by any method you like:

\[
\begin{align*}
\frac{V_S - V_a}{R_1} &= \frac{V_a - V_b}{R_2} + \frac{V_a}{R_4} \\
\frac{V_S - V_a}{R_1} &= \frac{V_a}{R_2} - \frac{I_S}{R_3} + \frac{V_a}{R_4} \\
\frac{V_a - V_b}{R_2} &= \frac{V_b}{R_3} + \frac{1}{R_2} + \frac{1}{R_3} \\
\frac{V_a - V_b}{R_2} &= \frac{V_a - V_b}{R_3} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \\
\frac{V_a - V_b}{R_2} &= \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \\
V_a &= 4.6 \cdot V \\
V_b &= -0.933 \cdot V
\end{align*}
\]

Or, with numbers

node a:
\[
\begin{align*}
360 \cdot \Omega \cdot \left(\frac{9 \cdot V - V_a}{40 \cdot \Omega}\right) &= \left(\frac{V_a - V_b}{120 \cdot \Omega} + \frac{V_a}{72 \cdot \Omega}\right) \cdot 360 \cdot \Omega \\
81 \cdot V - 9 \cdot V_a &= 3 \cdot V_a - 3 \cdot V_b + 5 \cdot V_a \\
81 \cdot V - 9 \cdot (1.5 \cdot V_b + 6 \cdot V) &= 3 \cdot (1.5 \cdot V_b + 6 \cdot V) - 3 \cdot V_b + 5 \cdot (1.5 \cdot V_b + 6 \cdot V) \\
81 \cdot V - 13.5 \cdot V_b - 54 \cdot V &= 4.5 \cdot V_b + 18 \cdot V - 3 \cdot V_b + 7.5 \cdot V_b + 30 \cdot V \\
81 \cdot V - 54 \cdot V - 18 \cdot V - 30 \cdot V &= -21 \cdot V = 4.5 \cdot V_b - 3 \cdot V_b + 7.5 \cdot V_b + 13.5 \cdot V_b = 22.5 \cdot V_b \\
V_b &= \frac{-21 \cdot V}{22.5} = -0.933 \cdot V
\end{align*}
\]

node b:
\[
\begin{align*}
240 \cdot \Omega \cdot \frac{V_a - V_b}{120 \cdot \Omega} &= \left(\frac{V_b - 0 \cdot V}{240 \cdot \Omega} + 50 \cdot \text{mA} \cdot 240 \cdot \Omega\right) \\
2 \cdot V_a - 2 \cdot V_b &= V_b + 48 \cdot \text{mA} \cdot 240 \cdot \Omega \\
V_a &= \frac{2 \cdot V_b + V_b + 12 \cdot V}{2} = 1.5 \cdot V_b + 6 \cdot V \\
\text{<-- substitute for } V_a
\end{align*}
\]

\[
\begin{align*}
V_a &= 1.5 \cdot V_b + 6 \cdot V = 4.6 \cdot V \\
V_a &= 1.5 \cdot V_b + 6 \cdot V = 4.6 \cdot V
\end{align*}
\]

Same as $V_L$ of Ex 4 of Thévenin examples:
Ex 3 Like Superposition Ex. 2

a) Use nodal analysis to find the voltage across $R_2$ ($V_{R2}$).

You **MUST** show all the steps of nodal analysis work to get credit, including drawing appropriate symbols and labels on the circuit shown.

1) Define ground and nodes:

2) Label unknown node voltages as $V_a$, $V_b$, ... and label the current in each resistor as $I_1$, $I_2$, ....

3) Write Kirchoff's current equations for each unknown node.

4) Replace the currents in the KCL equations with Ohm's law relationships.

5) Solve the equation for the unknown voltage.

$b)$ Find the current through $R_3$ ($I_{R3}$).

Remember, we needed to find the voltage across $R_2$ ($V_{R2}$).

$$V_{R2} = V_S - V_a = 7.2 \cdot V$$

$$I_{R3} = \frac{0 - V_a}{R_3} = -1.6 \cdot mA \text{  actually flows the other way}$$
Ex 4 Use nodal analysis to find the voltage across $R_5$ ($V_{R5}$) and the current through $R_1$ ($I_{R1}$). From exam 1, F09

You MUST show all the steps of nodal analysis work to get credit, including drawing appropriate symbols and labels on the circuit shown.

[node a:]

\[
I_{R1} + I_S = I_R + I_5
\]

\[
R_3 = 300 \cdot \Omega \quad \frac{V_{S1} - V_a}{R_1} + I_S = \frac{V_a - V_{S2}}{R_2 + R_3} + \frac{V_a - V_{S2}}{R_5}
\]

\[
\frac{12 \cdot V}{100 \cdot \Omega} - \frac{V_a}{100 \cdot \Omega} + 63 \cdot mA = \frac{V_a}{500 \cdot \Omega} - \frac{6 \cdot V}{500} + \frac{V_a}{600 \cdot \Omega} - \frac{6 \cdot V}{600}
\]

Multiply both sides by $3000 \cdot \Omega$

\[
360 \cdot V - 30 \cdot V_a + 189 \cdot V = 6 \cdot V_a - 36 \cdot V + 5 \cdot V_a - 30 \cdot V
\]

\[
360 \cdot V + 189 \cdot V + 36 \cdot V + 30 \cdot V = 6 \cdot V_a + 5 \cdot V_a + 30 \cdot V_a
\]

\[
615 \cdot V = 41 \cdot V_a
\]

\[
V_a := \frac{615 \cdot V}{41}
\]

\[
V_a = 15 \cdot V
\]

\[
V_{R5} = V_a - V_{S2} = 9 \cdot V
\]

\[
I_{R1} = \frac{V_{S1} - V_a}{R_1} = -30 \cdot mA
\]
What if one side of a voltage source isn't ground?

\[ I_1 + I_{VS2} = I_3 \]

\[ \frac{V_{S1} - V_a}{R_1} + ? = I_S \]

What do you put in for \( I_{VS2} \)?

Go to the other side of \( V_{S2} \).

\[ \frac{V_{S1} - V_a}{R_1} + \frac{0 - V_b}{R_2} = I_S \]

Only problem is that you get the same equation at node b!

Where does the second equation come from?

Use something like this: \( V_a = V_b + V_{S2} \)

Similar Circuit, but no \( V_{S1} \).

If the ground is already at the bottom, use the same method as above.

If you can choose your ground, you can make life a little simpler.
**DC Notes**

### Basic electrical quantities
- **Charge**, actually moves $Q$
- **Current**, like fluid flow $I = \frac{Q}{s}$
- **Voltage**, like pressure $V$
- **Resistance** $R = \frac{V}{I}$
- **Conductance** $G = \frac{1}{R}$
- **Power energy/time** $P = V \cdot I$

### Unit
- Coulomb (C)
- Amp (A, mA, µA,...)
- Volt (V, mV, kV,...)
- Ohm (Ω, kΩ, MΩ,...)
- Siemens (S, old unit mho)
- Watt (W, mW, kW, MW,...)
- Coulomb (C)

### Schematic symbols
- **Battery**
- **Current source**
- **Voltage sources**
- **Ideal wire**
- **Node**

### KCL, Kirchhoff’s Current Law
- $I_{in} = I_{out}$ of any point, part, or section

### KVL, Kirchhoff’s Voltage Law
- Voltage gains = Voltage drops around any loop

### Node
- All points connected by wire, all at same voltage (potential)

### Ohm’s law (resistors)
- $V = I \cdot R$
- $R = \frac{V}{I}$

### Power
- $P_{IN} = P_{OUT}$ for resistor circuits
- $P = V \cdot I$ for everything
- $\text{Contribute} = I^2 \cdot R = \frac{V^2}{R}$ for resistors

### Resistors
- **Series**: \( R_{eq} = R_1 + R_2 + R_3 + \ldots \)
- **Parallel**: \( R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots} \)

### Voltage divider:
- $V_{Rn} = V_{\text{total}} \cdot \frac{R_n}{R_1 + R_2 + R_3 + \ldots}$

### Current divider:
- $I_{Rn} = I_{\text{total}} \cdot \frac{R_n}{R_1 + R_2 + R_3 + \ldots}$

### Multiple unknowns:
1. Combine resistors into equivalents where possible.
2. Use superposition if there are multiple sources and you know all the resistors.
3. Use KCL, KVL, & Ohm’s laws to write multiple equations and solve.

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**ECE 2210 / 00**

A. Stoops
2/8/00, rev 1/28/06

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**Resistors**
- Series: \( R_{eq} = R_1 + R_2 + R_3 + \ldots \)
- Parallel: \( R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots} \)
### Thévenin equivalent

To calculate a circuit’s Thévenin equivalent:

1) Remove the load and calculate the open-circuit voltage where the load used to be. This is the Thévenin voltage ($V_{Th}$).
2) Zero all the sources. (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
3) Compute the total resistance between the load terminals. (DO NOT include the load in this resistance.) This is the Thévenin source resistance ($R_{Th}$).
4) Draw the Thévenin equivalent circuit and add your values.

### Norton equivalent

To calculate a circuit’s Norton equivalent:

1) Replace the load with a short (a wire) and calculate the short-circuit current in this wire. This is the Norton current ($I_N$). Remove the short.
2) Zero all the sources. (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
3) Compute the total resistance between the load terminals. (DO NOT include the load in this resistance.) This is the Norton source resistance ($R_N$). (Exactly the same as the Thévenin source resistance ($R_{Th}$)).
4) Draw the Norton equivalent circuit and add your values.

### OR (the more common way)...

1) Find the Thévenin equivalent circuit.
2) Convert to Norton circuit, $R_N = R_{Th}$ and $I_N = V_{Th}/R_{Th}$.

### Nodal Analysis

1) If the circuit doesn’t already have a ground, label one node as ground (zero voltage). If the ground can be defined as one side of a voltage source, that will make the following steps easier.
2) Label unknown node voltages as $V_a$, $V_b$, ... and label the current in each resistor as $I_1$, $I_2$, ....
3) Write Kirchhoff’s current equations for each unknown node.
4) Replace the currents in your KCL equations with expressions like the one below.

\[ I_1 = \frac{V_a - V_b}{R_1} \]

5) Solve the multiple equations for the multiple unknown voltages.

### Superposition

For circuits with more than 1 source.

1) Zero all but one source. (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
2) Compute your wanted voltage or current due to the remaining source. Careful, some may be negative.
3) Repeat the first two steps for all the sources.
4) Sum all the contributions from all the sources to find the actual voltage or current. **Watch your signs!**
AC stands for Alternating Current as opposed to DC, Direct Current. AC refers to voltages and currents that change with time, usually the voltage is + sometimes and - at other times. This results in currents with go one direction when the voltage is + and the reverse direction when the voltage is -.

AC is important for two reasons. Power is created and distributed as AC. Signals are AC.

**AC Power**

Power is generated by rotating magnetic fields. This naturally produces sinusoidal AC waveforms.

It is easier to make AC motors than DC motors.

**AC Power allows use of transformers to reduce line losses**

Transformers work with AC, but not DC. Transformers can be used to raise or lower AC voltages (with an opposite change of current). This can be very useful in power distribution systems. Power is voltage times current. You can distribute the same amount of power with high voltage and low current as you can with low voltage and high current. However, the lower the current, the lower the $I^2R$ losses in the wires (all real wires have some resistance). So you'd like to distribute power at the highest possible voltage. Transformers allow you to do this with AC, but won't work with DC.

Example:

**Without transformers**

Without transformers

**With transformers**

In this example, the power lost in the transmission lines is only 1/10,000th what it is without transformers. That's why they raise the voltage in transmission lines to the point where they crackle and buzz. That crackle is the sound of the losses into the surrounding air and can become significant if the voltage is too high.
Signals
A time-varying voltage or current that carries information. If it varies in time, then it has an AC component.

In some unpredictable fashion
DC is not a signal. Neither is a pure sine wave. If you can predict it, what information can it provide?
Neither DC nor pure sine wave have any "bandwidth". In fact, no periodic waveform is a signal & no periodic waveform has bandwidth. You need bandwidth to transmit information.

Signal sources
Microphone
Camera
Thermistor or other thermal sensor
Potentiometer
LVDT (Linear Variable Differential Transformer)
Light sensor
Computer
switch
etc...

A transducer is a device which transforms one form of energy to another. Some sensors are transducers, many are not

Most often a signal comes from some other system.

Periodic waveforms: Waveshape repeats
T = Period = repeat time
f = frequency, cycles / second \[ f = \frac{1}{T} = \frac{\omega}{2\pi} \]
\[ \omega = \text{radian frequency, radians/sec} \quad \omega = 2\pi f \]
A = amplitude
DC = average

Sinusoidal AC
y(t) = A \cdot \cos(\omega t + \phi)
voltage: \[ v(t) = V_p \cdot \cos(\omega t + \phi) \]
current: \[ i(t) = I_p \cdot \cos(\omega t + \phi) \]
Phase: \[ \phi = -\frac{\Delta t}{T} \cdot 360\text{-deg} \quad \text{or:} \quad \phi = -\frac{\Delta t}{T} \cdot 2\pi\text{-rad} \]

Other common periodic waveforms

All but the square and triangle waves have a DC component as well as AC.
Now that we have voltages and currents which can be functions of time, it's time to introduce the capacitor and the inductor.

**Fluid Model:**

\[ \begin{align*}
\text{voltage } v_C & \quad \text{like pressure } \Delta P \\
\text{flow} & \quad \text{is like current}
\end{align*} \]

**Electrical equivalent:**

\[
\begin{align*}
C & = \varepsilon \frac{A}{d} = \frac{Q}{V} = \frac{dq}{dv} \\
\text{units:} & \quad \text{farad} = \frac{coul}{volt} = \frac{amp\cdot sec}{volt}
\end{align*}
\]

\[ \begin{align*}
\mu F & = 1 \times 10^{-6} \quad \text{farad} \\
pF & = 1 \times 10^{-12} \quad \text{farad}
\end{align*} \]

For drawings of capacitors and info about tolerances, see Ch.3 of textbook.

**Basic equations you should know:**

\[
\begin{align*}
C & = \frac{Q}{V} \\
i_C & = C \frac{d}{dt} v_C
\end{align*}
\]

Energy stored in electric field:

\[ W_C = \frac{1}{2} C V_C^2 \]

**Capacitor voltage cannot** change instantaneously

- **parallel:**
  \[ C_{eq} = C_1 + C_2 + C_3 + \ldots \]

- **series:**
  \[ C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots} \]

Capacitors are the only "backwards" components.

**Sinusoids**

\[
\begin{align*}
i_C(t) & = I_p \cos(\omega t) \\
v_C(t) & = \frac{1}{C} \int i_C dt = \frac{1}{C \omega} I_p \sin(\omega t) = \frac{1}{C \omega} I_p \cos(\omega t - 90^\circ)
\end{align*}
\]

Voltage "lags" current, makes sense, current has to flow in first to charge capacitor.

**Steady-state or Final conditions**

If a circuit has been connected for "a long time", then it has reached a steady state condition. That means the currents and voltages are no longer changing.

\[
\begin{align*}
\frac{d}{dt} v_C & = 0 \\
i_C & = C \frac{d}{dt} v_C = 0
\end{align*}
\]

no current means it looks like an open
Example
The voltage across a 0.5 μF capacitor is shown below. Make an accurate drawing of the capacitor current. Label the y-axis of your graph (I’ve already done the time-axis).

The accuracy of your plot at 0, 2, 6, and 8 ms is important, so calculate those values and plot or label them carefully. Between those points your plot must simply be the correct shape.

\[ C = 0.5 \, \mu \text{F} \]

\[ \text{The curve is 2}^{\text{nd}} \text{ order} \]

\[ v_C(t) \]

\[ i_C(t) \]

1 - 2ms: \[ i_C = C \frac{\Delta V}{\Delta t} = 0.5 \, \mu \text{F} \cdot \frac{4 \, \text{V}}{2 \, \text{ms}} = -1 \, \text{mA} \]

2ms - 6ms: \[ \text{Initial slope is zero and the final slope is positive, so the current must be a triangle that starts at zero and ends at some height.} \]

\[ \Delta v_C(t) = \frac{1}{C} \int_0^t i_C(t) \, dt \]

\[ 8 \cdot \text{V} = \frac{1}{C} \left[ \frac{4 \, \text{ms} \cdot \text{height}}{2} \right] \]

\[ \text{height} = 8 \cdot \text{V} \frac{C \cdot 2}{4 \, \text{ms}} = 2 \, \text{mA} \]

6ms - 8ms: \[ \text{Slope is zero, so the current must be zero.} \]

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Fluid Model:

Basic equations you should know:

\[ v_L = L \frac{di_L}{dt} \]

Energy stored in electric field: \[ W_L = \frac{1}{2} L I_L^2 \]

Inductor current cannot change instantaneously

Units: henry = \[ \frac{\text{volt} \cdot \text{sec}}{\text{amp}} \]

\[ \text{mH} = 10^{-3} \cdot \text{H} \quad \mu \text{H} = 10^{-6} \cdot \text{H} \]
\[ L_{eq} = \frac{1}{L_1 + \frac{1}{L_2 + \frac{1}{L_3 + \ldots}}} \]

Sinusoids
\[ i_L(t) = I_p \cdot \cos(\omega t) \]
\[ v_L(t) = L \cdot \frac{d}{dt} i_L = L \cdot \omega \cdot \left( I_p \cdot \sin(\omega t) \right) = L \cdot \omega \cdot I_p \cdot \cos(\omega t + 90\text{-deg}) \]

Voltage "leads" current, makes sense, voltage has to present to make current change, so voltage comes first.

Resonance
Series resonance
Parallel resonance
looks like a short at resonance frequency
looks like an open at resonance frequency

The resonance frequency is calculated the same way for either case:
\[ \omega_o = \frac{1}{\sqrt{L \cdot C}} \text{(rad/sec)} \quad \text{OR..} \quad \omega_o = \frac{1}{\sqrt{L_{eq} \cdot C_{eq}}} \]

If you have multiple capacitors or inductors which can be combined:
\[ f_o = \frac{\omega_o}{2 \pi} \text{ (Hz)} \]

Steady-state of Final conditions
If a circuit has been connected for "a long time", then it has reached a steady state condition. That means the currents and voltages are no longer changing.
\[ \frac{d}{dt} i_L = 0 \quad v_L = L \frac{d}{dt} i_L = 0 \]
No voltage means it looks like a short.

Examples
Ex 1
Find the resonant frequency (or frequencies) of the circuit shown (in cycles/sec or Hz).
\[ \omega_o = \frac{1}{\sqrt{L_{eq} \cdot C_{eq}}} \quad \omega_o = 11547 \text{ rad/sec} \quad f_o = \frac{\omega_o}{2 \pi} = 1838 \text{ Hz} \]
Ex 2

The current through a 0.3mH inductor is shown below. Make an accurate drawing of the inductor voltage. Make reasonable assumptions where necessary. Label your graph.

\[
L := 0.3 \cdot \text{mH}
\]

The curve is 2nd order and ends at 8\mu s

0 - 2\mu s: No change in current, so: \( v_L = 0 \)

2\mu s - 4\mu s: \[ v_L = \frac{\Delta i}{\Delta t} = \frac{0.3 \cdot \text{mH} \cdot 0.6 \cdot \text{A}}{2\mu s} = -90 \cdot \text{V} \]

4\mu s - 8\mu s: Initial slope is positive and the final slope is zero, so the voltage must be a triangle that starts at some height and ends at zero.

\[
\Delta i_L(t) = \frac{1}{L} \int_{0}^{t} v_L(t) \, dt
\]

6\cdot mA = \frac{1}{0.3 \cdot \text{mH}} \left( \frac{4\mu s \cdot \text{height}}{2} \right)

height = \frac{0.6 \cdot \text{A} \cdot 0.3 \cdot \text{mH} \cdot 2}{4\mu s} = 90 \cdot \text{V}

8\mu s - 10\mu s: No change in current, so: \( v_L = 0 \)

Ex 3

Given a voltage, find the current, \( L := 4 \cdot \text{mH} \)

\[
\Delta i_L(t) = \frac{1}{L} \left[ \int_{0}^{2\mu s} 20 \cdot \text{V} \, dt = 5 \cdot \text{mA} \right. \\
- \int_{1\mu s}^{8\mu s} -10 \cdot \text{V} \, dt + 5 \cdot \text{mA} = -5 \cdot \text{mA} \\
\left. \int_{4\mu s}^{10\mu s} V(t) \, dt + -5 \cdot \text{mA} \right]
\]

Ex 4

The following circuit has been connected as shown for a long time. Find the energy stored in the capacitor and the inductor.

\[
\frac{V}{2} \cdot \frac{2}{R_1} = \frac{V}{2} \cdot 4 \cdot \Omega = 0 \cdot \text{mA} \quad \text{etc...}
\]
Capacitors
\[ C = \frac{Q}{V} \text{ farad} = \frac{\text{coul}}{\text{volt}} = \frac{\text{amp-sec}}{\text{volt}} \]
\[ v_C = \frac{1}{C} \int_{0}^{t} i_C \, dt \quad \text{initial voltage} \]
\[ i_C = \frac{1}{C} v_C(0) \]
\[ i_C = C \frac{d}{dt} v_C \]
Energy stored in electric field: \[ W_C = \frac{1}{2} C \cdot V_C^2 \]
Parallel: \[ C_{eq} = C_1 + C_2 + C_3 + \ldots \]
Series: \[ C_{eq} = \frac{1}{C_1 + C_2 + C_3 + \ldots} \]
Steady-state sinusoids:
Impedance: \[ Z_C = \frac{1}{j \omega C} = -\frac{j}{\omega C} \]
Current leads voltage by 90 deg
Inductors
\[ h = \frac{\text{volt-sec}}{\text{amp}} \]
\[ i_L = \frac{1}{L} \int_{-\infty}^{t} v_L \, dt \]
\[ v_L = \frac{1}{L} \int_{0}^{t} i_L(0) \, dt \]
Energy stored in magnetic field: \[ W_L = \frac{1}{2} L I_L^2 \]
Series: \[ L_{eq} = L_1 + L_2 + L_3 + \ldots \]
Parallel: \[ L_{eq} = \frac{1}{L_1 + L_2 + L_3 + \ldots} \]
Steady-state sinusoids:
Impedance: \[ Z_L = j \omega L \]
Current lags voltage by 90 deg
RC and RL first-order transient circuits
For all first order transients:
\[ v_X(t) = v_X(\infty) + (v_X(0) - v_X(\infty)) e^{-\frac{t}{\tau}} \]
\[ i_X(t) = i_X(\infty) + (i_X(0) - i_X(\infty)) e^{-\frac{t}{\tau}} \]
Find initial Conditions \((v_C \text{ and/or } i_L)\)
Find conditions just before time \(t = 0\), \(v_C(0-)\) and \(i_L(0-)\). These will be the same just after time \(t = 0\), \(v_C(0+)\) and \(i_L(0+)\) and will be your initial conditions. (If initial conditions are zero: Capacitors are shorts, Inductors are opens.)
Use normal circuit analysis to find your desired variable: \(v_X(\infty)\) or \(i_X(\infty)\)
Find final conditions (*steady-state* or *forced* solution)
Inductors are shorts  Capacitors are opens  Solve by DC analysis \(v_X(\infty)\) or \(i_X(\infty)\)

RC Time constant = \(\tau = RC\)

RL Time constant = \(\tau = \frac{L}{R}\)

Curves
\[ v_C(t) \]
\[ i(t)R = v_R(t) \]
\[ v_L(t) \]
\[ i(t)R = v_R(t) \]
\[ e^{-1} = 0.368 \]
\[ 1 - e^{-1} = 0.632 \]
1.1 Introduction

Transient: A transient is a transition from one state to another. If the voltages and currents in a circuit do not change with time, we call that a "steady state". In fact, as long as the voltages and currents are steady AC sinusoidal values, we can call that a steady state as well. Up until now we've only discussed circuits in a single steady state. But what happens when the state of a circuit changes, say from "off" to "on"? Can the state of the circuit change instantaneously? No, nothing ever changes instantaneously, the circuit state will go through some transition from the initial state, "off" to the final state, "on" and that change will take some amount of time. The same is true in mechanical systems. If you want to change the velocity of a mass or the level of fluid in a tank or the temperature of your coffee, that transition from one state to another will take some time.

The drawings on this page show some typical transients that can occur when a circuit is first turned on. The initial state of all the waveforms is 0. The final state is either 1 or a sine wave with an amplitude of 1. Notice that in all four cases the transient effects decay exponentially and that all four waveforms have pretty nearly reached their steady-state values by the end of the graph.

Transient analysis: Needless to say, the analysis of these transients is a bit more involved than the steady state. In fact, it usually involves two steady state analyses just to find the initial and final states of the circuit, and then you still need to figure out what happens in between.

Transients are not instant because capacitors and inductors in the circuit store energy, and moving the energy into or out of these parts takes some time. The voltage-current relationships of capacitors and inductors are differential equations, so transient analysis will involve solving differential equations. But don't panic, you'll learn some nice tricks and techniques for dealing with these equations—tricks and techniques that you can use in any engineering field, not just EE. Actually, all that phasor stuff you used with AC circuits was also a trick to simplify the differential equations, unfortunately, that trick only works for sinusoids in steady state.

DC circuits with only resistors also experience transients, but these are due to non-ideal capacitance and inductance of the parts and wires that we haven't considered before. These transients happen so fast that we won't worry about them.
Importance: So why are transients important?
Two reasons really. DC and steady-state AC are fine for moving and using electrical power, but sometimes you need to turn them on and off and you may need to know what happens at those times. That need turns out to be relatively rare and probably couldn’t justify the time we’ll spend studying transients. It’s signals processing and control systems really drive our study of transients.

Signals are electrical voltages and currents that carry information. The information could be audio or video or the information might be about the position or speed of mechanical parts, or about the temperature or level of fluids or chemicals or practically anything you can imagine. To carry information signals have to change in some way that we can’t predict and we’ll need to have some idea how a circuit will respond to those changes. Changes are transients. However, since these changes can’t be known beforehand we usually find a circuit's response to specific types of inputs and then draw conclusions about the effectiveness or stability in the general case. Often the electrical circuit is just one part of a larger system that may include mechanical, hydraulic, or thermal systems. See box.

1.2 First-order transients

Analysis of a circuit with only one capacitor or one inductor results in a first-order differential equation and the transients are called first-order transients.

Series RC circuit, traditional way: Look at the circuit at right. It shows a capacitor and a resistor connected to a voltage source by way of a switch that is closed at time t=0. Before the switch is closed the current i(t) and the voltage v_R are both 0, but the voltage v_C is unknown. Remember a capacitor is capable of storing a charge, so we don’t know what its charge might be unless we or can measure it or its is given. I’ll call it the initial voltage (v_C(0)). Because the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor just after the switch closes must be the same as it was just before the switch closes.

Now we just have to apply the basic circuit laws

\[ V_{in} = v_R + v_C \]

\[ i \cdot R + v_C = i \cdot R + \frac{1}{C} \int_{\infty}^{t} i_C dt \]

Making the obvious substitution.

The next step here would be to differential both sides of the equation, but if you’re a little more clever, there’s an easier way, check this out:

Make this substitution instead

\[ i = i_C = C \frac{dv_C}{dt} \]

to get

\[ V_{in} = R \cdot C \frac{dv_C}{dt} + v_C \]

Waa-laah, no integration. Always try to write your differential equations without integrals, it will eliminate one more source of mistakes. We now have a differential equation in terms of v_C. If v_R isn’t the variable we want to find in our analysis then we can always go back to the circuit later and find the current or voltage v_R by simple circuit analysis after we’ve found v_C.

Transients p. 1.2
So now we have to solve the differential equation. Recall from your differential equations class that first order differential equations are always solved by equations of the following form.

Standard first order differential equation answer: \( v_C(t) = A + B \cdot e^{t} \)

And, by differentiation: \( \frac{dv_C}{dt} = B \cdot s \cdot e^{t} \)

Substitute these back into the original equation:

\[
V_{in} = R \cdot C \cdot \frac{dv_C}{dt} + v_C = R \cdot C \cdot B \cdot s \cdot e^{t} + \left( A + B \cdot e^{t} \right) = R \cdot C \cdot B \cdot s \cdot e^{t} + B \cdot e^{t} + A
\]

We can separate this equation into two parts, one which is time dependent and one which is not. Each part must still be an equation.

Time independent (forced) part: \( V_{in} = A \), \( A = V_{in} = \text{final condition} = v_C(\infty) \)

Time dependent (transient) part: \( 0 = R \cdot C \cdot B \cdot s \cdot e^{t} + B \cdot e^{t} \),

Divide both sides by \( B \cdot e^{t} \) to get \( 0 = R \cdot C \cdot s + 1 \), \( s = \frac{1}{R \cdot C} = \frac{1}{\tau} \), where \( \tau = R \cdot C \)

This \( \tau \) is called the "time constant" and will become a rather important little character.

Put the parts we know back into the expression for \( v_C(t) = V_{in} + B \cdot e^{-\frac{t}{R \cdot C}} = v_C(\infty) + B \cdot e^{-\frac{t}{R \cdot C}} \)

at time \( t = 0 \): \( v_C(0) = V_{in} + B \), \( B = v_C(0) - V_{in} = v_C(0) - v_C(\infty) \) \( B \) is the difference between \( v_C \) at the start and \( v_C \) at the end.

And finally: \( v_C(t) = V_{in} + B \cdot e^{-\frac{t}{R \cdot C}} = v_C(\infty) + \left( v_C(0) - v_C(\infty) \right) \cdot e^{-\frac{t}{R \cdot C}} \)

It turns out that all first-order transient solutions will have the same form, just different variables and time constants.

Once you have \( v_C(t) \), you can also find \( v_R(t) \) and/or \( i(t) \) from \( v_C(t) \) if you want.

\[
v_R(t) = V_{in} + v_C(t) = V_{in} - \left( B \cdot e^{-\frac{t}{R \cdot C}} + V_{in} \right) = -B \cdot e^{-\frac{t}{R \cdot C}} = -B \cdot e^{-\frac{t}{\tau}} = \left( v_C(0) - v_C(\infty) \right) \cdot e^{-\frac{t}{R \cdot C}}
\]

\[
i(t) = C \cdot \frac{dv_C}{dt} = -C \cdot B \cdot \frac{1}{R \cdot C} \cdot e^{-\frac{t}{R \cdot C}} = \frac{B}{R} \cdot e^{-\frac{t}{\tau}} = \frac{\left( v_C(0) - v_C(\infty) \right)}{R} \cdot e^{-\frac{t}{R \cdot C}}
\]

Let's plot these and see what they actually look like. These graphs show the capacitor charging from it's initial value to \( V_{in} \) and \( v_R \) falling to \( 0 \) (same for \( i_R \)).

The curves are generalized based on the concept of the time constant, which is why we introduced the time constant. Later we'll look at these kind of curves in greater detail.

Ok, that was fun, but you might ask at this point if there isn't an easier way. Yes, in fact, there is. We'll look at next.
First-Order Transients the Easy Way

Notice in the preceding analysis that I made a very standard guess at the solution of the differential equation.

Standard first order differential equation answer:
\[ v_C(t) = A + B \cdot e^{-t} \]

Further notice that A turned out to be the final condition and that B turned out to be the difference between the initial and final conditions. Finally, remember that I renamed s to -1/τ. All of this can be generalized to any first order system. The answer will always be in this form:
\[
\begin{align*}
\text{final condition} & \quad \text{time constant} \\
\text{initial condition} & \\
\end{align*}
\]

For all first order transients:
\[ x(t) = x(\infty) + \left( x(0) - x(\infty) \right) e^{-t/\tau} \]

x(t) could be any variable in any first-order system. It could be a temperature, or a fluid level, or a velocity, but for us it usually means voltages and currents, so we'll have solutions like these.

\[
\begin{align*}
 v_X(t) & = v_X(\infty) + \left( v_X(0) - v_X(\infty) \right) e^{-t/\tau} \\
i_X(t) & = i_X(\infty) + \left( i_X(0) - i_X(\infty) \right) e^{-t/\tau} 
\end{align*}
\]

You find Initial and final conditions from steady-state analysis. That leaves only one thing that you have to find from the differential equation-- the time constant. If we could only figure out what the time constant of a circuit (or system) is, then we could almost jump straight to the solution.

The first way to find the time constant is to simply remember it's form for a few cases, like the for RC circuit. Even if the circuit doesn't look exactly like the standard RC series circuit, Thévenin can help us make it look that way. Since nearly all of our first order circuits will involve a single capacitor or a single inductor this is not an impractical method at all.

Another way to find the time constant is to manipulate the differential equation into this particular form
\[ \text{constant} = X + \tau \frac{dX}{dt} \quad \text{with no factor in front of the "X" term.} \quad \text{Whatever the factor in front of} \quad \frac{dX}{dt} \quad \text{turns out to be, that will be} \quad \tau. \]

For the RC circuit the differential equation could be written as
\[ V_{\text{in}} = R \cdot C \cdot \frac{d}{dt} v_C + v_C \quad \text{notice that the factor in front of} \quad \frac{d}{dt} v_C \quad \text{is indeed} \quad \tau. \]

Finally, there is an even easier way based on the LaPlace "s" and s-impedances that we can use in circuits and equations in place of differentials and integrals. You'll see this last method later, after second-order transients. (Incidentally, this is the reason that I chose to use an s as the unknown in the exponential.)

Series RL circuit: OK, if it's so easy, let's try it with a series RL circuit.
\[ V_{\text{in}} = v_R + v_L \quad V_{\text{in}} = i \cdot R + L \cdot \frac{d}{dt} i \quad \frac{V_{\text{in}}}{R} = i + \frac{L}{R} \cdot \frac{d}{dt} i \]

So, the time constant must be \( \tau = \frac{L}{R} \) That wasn't too bad.

Initial condition: \( i_L(0) = 0 \) If the switch was initially open the the current just before the switch was closed was 0, and inductor current can't change instantly.

Final condition: \( i_L(\infty) = \frac{V_{\text{in}}}{R} \) The inductor looks like an short for steady-state DC.

So:
\[ i_L(t) = i_L(\infty) + \left( i_L(0) - i_L(\infty) \right) e^{-t/\tau} = \frac{V_{\text{in}}}{R} + \left( 0 - \frac{V_{\text{in}}}{R} \right) e^{-t/\tau} = \frac{V_{\text{in}}}{R} \left( 1 - e^{-R \cdot t} \right) \]

Well, that's wasn't too painfull, was it?

Transients p. 1.4
1.3 Initial and Final Conditions

More than once I’ve said that the initial and final conditions are found from steady-state analysis of the circuit. It’s about time I said how.

Initial Conditions: There are two very important concepts that you use to find the initial conditions.

1) Capacitor voltage cannot change instantaneously, $v_C(0+) = v_C(0-)$. If you can find the capacitor voltage just before time $t = 0$ (or whatever starts the transient), then you know what it is just after time $t = 0$, $v_C(0+) = v_C(0-)$. It cannot change instantaneously. Often you’ll use the methods outlined below to find the final condition of the previous circuit, especially if the circuit's been in that condition for "a long time". Sometimes you’ll have to solve the previous transient to find the initial condition for the next transient.

If you cannot find the capacitor voltage just before time $t = 0$ from the circuit, then you'll have to be told what the initial voltage or charge is. Capacitors can hold a charge for a long time, and can be moved from one circuit to another without losing the charge. High school electronics students like to charge capacitors and leave them where they'll shock some poor unsuspecting soul. Of course you'd never do something as childish as that. Occasionally you may be told what the initial charge is in terms of coulombs. In that case remember the definition of capacitance.

$$C = \frac{Q}{V}$$

which can be rearranged to

$$V = \frac{Q}{C}$$

If you have nothing else to go on, assume the initial voltage is 0.

2) Inductor current cannot change instantaneously, $i_L(0+) = i_L(0-)$. If you can find the inductor current just before time $t = 0$ (or whatever starts the transient), then you know what it is just after time $t = 0$, $i_L(0+) = i_L(0-)$. It cannot change instantaneously.

If you cannot find the inductor current just before time $t = 0$ from the circuit, then assume it's 0. Real circuits and real inductors always have some resistance so inductor currents just don't last very long (unless you're dealing with superconductors). Inductors would be very difficult to move from one circuit to another without losing the current. If you're given an initial current for a problem, realize that this is probably just to make the problem more interesting, or the initial current comes from previous analysis.

Do not mix these two concepts up. Capacitor current and inductor voltage can both change instantaneously with no problem at all.

Final Conditions: This is steady-state analysis. The steady-state is the final condition.

DC sources

If all the voltage and current sources are DC, then at the final condition the capacitors are all done charging so $i_C = 0$, and you can treat them as open circuits. When you find the voltage across the open, that will be the final capacitor voltage. You've done this sort of thing before to find the energy stored in a capacitor.

Replace capacitors with opens

Replace inductors with wires

At the final condition the inductor currents are also no longer changing, so the voltage across an inductor is 0. Treat inductors as wires (short circuits). When you find the current through the wire, that will be the final inductor current.

AC sources

Use phasor analysis ($j\omega$). Remember that phasor analysis was also called "steady-state AC". One of the primary assumptions was that the transients had all died out.

Transients p. 1.5
1.4 Exponential Curves

Before we go on to second-order transients we should take a closer look at some of the characteristics of exponential curves. The curves that show up as answers to our transient problems are shown below. The transient effects always die out after some time, so the exponents are always negative. Just think about what a positive exponent would mean. That wouldn't be a transient-- that would be exponential growth, like the population.

Some important features:

1) These curves proceed from an initial condition to a final condition. If the final condition is greater than the initial, then the curve is said to be a "rising" exponential. If the final condition is less than the initial, then the curve is called a "decaying" exponential.

2) The curves' initial slope is $\pm \frac{1}{\tau}$. If they continued at this initial slope they'd be done in one time constant.

3) In the first time constant the curve goes 63% from initial to the final condition.

4) After three time constants the curve is 95% of the way to the final condition.

5) By five time constants the curve is within 1% of the final condition and is usually considered finished. Mathematically, the curve approaches the final condition asymptotically and never reaches it. In reality, of course, this is nonsense. Whatever difference there may be between the mathematical solution and the final condition will soon be overshadowed by random fluctuations (called noise) in the real circuit.

Transients p. 1.6
Ex1  a) Find the expression for \( v_c(t) \) if the switch is closed at time \( t = 0 \) and \( v_c(0) = 0 \).

\[
\begin{align*}
v_c(t) &= v_c(\infty) + \left( v_c(0) - v_c(\infty) \right) e^{-\frac{t}{\tau}} \\
\tau &= R \cdot C \\
&= 60 \, \mu s \\
v_c(\infty) &= 9 \, V \\
v_c(t) &= 9 \, V - (9 \, V - 0 \, V) \cdot e^{-\frac{t}{\tau}} \\
&= \frac{9 \, V}{R} \cdot e^{-\frac{t}{\tau}}
\end{align*}
\]

b) What is the voltage across the capacitor, \( C \), at \( t = 0.1 \, ms \)?

\[
v_c(25 \, \mu s) = 9 \, V - 9 \, V \cdot e^{-\frac{100 \, \mu s}{60 \, \mu s}} = 7.3 \, V
\]

c) When will the current through the resistor be \( i_R = 5 \, mA \)?

\[
i_R(\infty) = 0 \, mA \\
i_R(0) = \frac{9 \, V}{R} = 15 \, mA \\
i_R(t) = i_R(\infty) + \left( i_R(0) - i_R(\infty) \right) e^{-\frac{t}{\tau}} \\
&= 0 \, mA + (15 \, mA - 0 \, mA) e^{-\frac{t}{\tau}} \\
&= 10.976 \, mA \cdot e^{-\frac{t}{60 \, \mu s}} = 5 \, mA \, \text{ at some time, } t
\]

Solve for \( t = -\tau \ln \left( \frac{5 \, mA}{15 \, mA} \right) = 65.92 \, \mu s \)

d) When will the current through the resistor be \( i_R = 20 \, mA \)?

Since the initial condition is about 15mA and the final condition is 0mA, \( i_R \) will never be 20mA.

Ex2  A 1000 \( \mu F \) capacitor has an initial charge of 12 volts. A 20-\( \Omega \) resistor is connected across the capacitor at time \( t = 0 \). Find the energy dissipated by the resistor in the first 5 time constants.

After 5 time constants nearly all of the energy initially stored in the capacitor will be dissipated by the resistor.

\[
C := 1000 \, \mu F \\
V_C := 12 \, V \\
W_C := \frac{1}{2} C \cdot V^2 \\
W_C = 0.072 \, J
\]

You can get to this answer just by knowing a little about the exponential curve, but what if you want a more accurate answer? Then you'll have to find the remaining voltage across the capacitor at \( t = 5t \) and subtract the energy left in the capacitor at that time.

\[
v_C(0) = 12 \, V \\
v_C(\infty) = 0 \, V \\
v_C(t) = v_c(\infty) + \left( v_c(0) - v_c(\infty) \right) e^{-\frac{t}{\tau}} \\
&= 0 \, V + (12 \, V - 0 \, V) \cdot e^{-\frac{t}{\tau}} \\
&= 12 \, V \cdot e^{-\frac{t}{\tau}}
\]

at \( t = 5t \):

\[
v_C(5t) = 12 \, V \cdot e^{-5} = 81 \, mV \\
\frac{1}{2} C \cdot (81 \, mV)^2 = 3.281 \times 10^{-6} \, J
\]

Not surprisingly, this makes no significant difference:

\[
W_R = W_C - \frac{1}{2} C \cdot (81 \, mV)^2 = 0.072 \, J
\]
Ex3 The capacitor is initially uncharged. The switch is in the upper position from 0 to 2ms and is switched down at time $t = 2ms$.

a) What is the capacitor voltage, $v_C(t)$

First interval $v_C(0) = 0 \cdot V$

$v_C(\infty) = 24 \cdot V$

$v_C(t) = v_C(\infty) + \left( v_C(0) - v_C(\infty) \right) \cdot e^{-\frac{t}{\tau}} = 24 \cdot V + (0 \cdot V - 24 \cdot V) \cdot e^{-\frac{t}{1.08 \cdot ms}}$

at 2ms: $24 \cdot V - 24 \cdot V \cdot e^{-\frac{2 \cdot ms}{1.08 \cdot ms}} = 20.23 \cdot V$

Second interval, define a new time, $t' = t - 2ms$

$v_C(t') = v_C(\infty) + \left( v_C(0) - v_C(\infty) \right) \cdot e^{-\frac{t'}{\tau'}} = 10 \cdot V + (20.23 \cdot V - 10 \cdot V) \cdot e^{-\frac{t'}{0.96 \cdot ms}} = 10 \cdot V + 10.23 \cdot V \cdot e^{-\frac{0.96 \cdot ms}{2 \cdot ms}}$

0 < $t < 2ms$

$V_C(t) = 24 \cdot V - 24 \cdot V \cdot e^{-\frac{1.08 \cdot ms}{1.08 \cdot ms}}$

$t > 2ms$

$V_C(t) = 10 \cdot V + 10.23 \cdot V \cdot e^{-\frac{0.96 \cdot ms}{2 \cdot ms}}$

b) When is voltage across the capacitor 12V AND getting smaller?

$12 \cdot V = 10 \cdot V + 10.233 \cdot V \cdot e^{-\frac{0.96 \cdot ms}{2 \cdot ms}}$

$\frac{12 \cdot V - 10 \cdot V}{10.23 \cdot V} = e^{-\frac{12 \cdot V - 10 \cdot V}{10.23 \cdot V}} = -\frac{12 \cdot V}{10.23 \cdot V} \cdot \ln\left(\frac{12 \cdot V - 10 \cdot V}{10.23 \cdot V}\right) = -\frac{12 \cdot V}{10.23 \cdot V} \cdot \ln\left(\frac{12 \cdot V - 10 \cdot V}{10.23 \cdot V}\right)$

$t_{12} = -0.96 \cdot ms \ln\left(\frac{12 \cdot V - 10 \cdot V}{10.23 \cdot V}\right) = 1.57 \cdot ms$

$2 \cdot ms + 1.57 \cdot ms = 3.57 \cdot ms$
Ex4  a) Find the complete expression for $i_L(t)$.

Before the switch closes, $t = 0^{-}$

$$R_{Th} = \frac{\frac{1}{R_1} + \frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + R_3} = 30\,\Omega$$

$$\tau = \frac{L}{R_{Th}} = 100\,\mu s$$

$$i_L(t) = i_L(\infty) + \left( i_L(0) - i_L(\infty) \right) e^{-t/\tau} = 375\,mA + (0\,mA - 375\,mA) e^{-t/100\mu s} = 375\,mA - 375\,mA e^{-t/100\mu s}$$

b) When is the voltage across $R_2 = 10V$?

Before the switch closes, $t = 0^{-}$

$$v_{R2}(0) = \frac{R_2}{R_1 + R_2} V_{in} = 11.25\,V$$

$$v_{R2}(\infty) = v_{R3}(\infty) = \frac{R_2}{R_1 + R_2 + R_3} V_{in} = \frac{R_2}{R_1 + R_2 + R_3} 15.0\,V = 5.625\,V$$

$$v_{R2}(t) = v_{R2}(\infty) + \left( v_{R2}(0) - v_{R2}(\infty) \right) e^{-t/\tau} = 5.625\,V + (11.25\,V - 5.625\,V) e^{-t/100\mu s} = 10\,V \text{ at some time, solving for that time...}$$

$$t = -\tau \ln \left( \frac{10\,V - 5.625\,V}{11.25\,V - 5.625\,V} \right) = 25\,\mu s$$

Alternatively, when $v_{R2}(t) = 10V$, then $v_{R1}(t) = 5V$ and $i_L(t) = \frac{5\,V - 10\,V}{R_1 - \frac{10\,V}{R_2}} = 83.333\,mA$

$$t = -\tau \ln \left( \frac{83.333\,mA - 375\,mA}{-375\,mA} \right) = 25\,\mu s$$

c) What is the $v_L(t)$ expression?

$$v_L(t) = v_L(\infty) + \left( v_L(0) - v_L(\infty) \right) e^{-t/\tau} = 0\,V + (11.25\,V - 0\,V) e^{-t/100\mu s}$$
Ex5 The switch has been closed for a long time and is opened (as shown) at time \( t = 0 \).

a) Find the complete expression for \( i_L(t) \).

Before the switch opens, \( t = 0^- \):

\[
i_L(0) = I_S \frac{R_3}{R_1 + R_2 + R_3} = 50 \text{mA}
\]

Final time, \( t = \infty \):

\[
i_L(\infty) = I_S \frac{R_3}{R_1 + \frac{1}{R_3}} = 112.5 \text{mA}
\]

b) Find \( i_L \) at time \( t = 1.4 \tau \).

\[
i_L(1.4 \tau) = 112.5 \text{mA} - 62.5 \text{mA} e^{\frac{-1.4 \tau}{120.4 \mu s}} = 97.088 \text{mA}
\]

c) At time \( t = 1.4 \tau \) the switch is closed again. Find the complete expression for \( i_L(t') \), where \( t' \) starts at \( t = 1.4 \tau \).

Be sure to clearly show the time constant.

\[
i_L(0) = 97.1 \text{mA} \text{ from part b)}
\]

\[
i_L(\infty) = 50 \text{mA} \text{ initial value from part a)}
\]

\[
i_L(t') = i_L(\infty) + (i_L(0) - i_L(\infty)) e^{\frac{-t'}{108.3 \mu s}} = 50 \text{mA} + (97.1 \text{mA} - 50 \text{mA}) e^{\frac{-t'}{108.3 \mu s}} = 50 \text{mA} + 47.1 \text{mA} e^{\frac{-t'}{108.3 \mu s}}
\]
**Complex Numbers**

### Rectangular Form

- **$A = a + bj$**
- **Re($A$) = $a$**
- **Im($A$) = $b$**

### Polar Form

- **$A = A \cdot e^{j \theta}$**
- **Re($A$) = $A \cdot \cos(\theta)$**
- **Im($A$) = $A \cdot \sin(\theta)$**

### Conversions

- **$A = |A| = \sqrt{a^2 + b^2}$**
- **$\theta = \text{arg}(A) = \tan^{-1} \left( \frac{b}{a} \right)$**
- **$a = A \cdot \cos(\theta)$**
- **$b = A \cdot \sin(\theta)$**

### Special Cases

- **$j : = \sqrt{-1} = e^{j90\deg}$**
- **$\frac{1}{j} = -e^{-j90\deg} = e^{-j0\deg} = 1$**
- **$e^{-j180\deg} = e^{j-180\deg} = -1$**
- **$j \cdot e^{j\theta} = e^{j(\theta + 90\deg)}$**

Define a 2nd number:

- **Rectangular: $D = c + dj$**
- **Polar: $D = D \cdot e^{j\theta}$**

### Equality

- **$A = D$ if and only if $a = c$ and $b = d$ OR $A = D$ and $\theta = \phi$**

### Addition and Subtraction

- **$A + D = (a + bj) + (c + dj) = (a + c) + (b + d)j$**
- **$A - D = (a + bj) - (c + dj) = (a - c) + (b - d)j$**

### Multiplication and Division

- **$A \cdot D = (a + bj) \cdot (c + dj) = (a \cdot c - b \cdot d) + (b \cdot c + a \cdot d)j$**

#### Rectangular:

- **$A = \frac{a + bj}{c + dj}$**
- **$D = \frac{a - bj}{c - dj}$**

#### Polar:

- **$A \cdot D = A \cdot e^{i\theta} \cdot D \cdot e^{i\phi} = A \cdot D \cdot e^{i(\theta + \phi)}$**
- **$A = \frac{A \cdot e^{i\theta}}{D \cdot e^{i\phi}} = A \cdot e^{i(\theta - \phi)}$**

### Powers

- **$A^n = A^n \cdot e^{j(n \cdot \theta)} = A^n \cdot \cos(n \cdot \theta) + A^n \cdot \sin(n \cdot \theta) \cdot j$**

### Conjugates

- **$A = a + bj$**
- **$\overline{A} = a - bj$**
- **$A = A \cdot e^{j\theta}$**
- **$\overline{A} = A \cdot e^{-j\theta}$**

### Euler's equation

- **$e^{j\alpha} = \cos(\alpha) + j \cdot \sin(\alpha)$**
- **$e^{j(\omega t + \theta)} = \cos(\omega t + \theta) + j \cdot \sin(\omega t + \theta)$**

### Calculus

Remember, when we write $e^{j\theta}$, we really mean $e^{j(\omega t + \theta)}$

- **$\frac{d}{dt} A = \frac{d}{dt} (A \cdot e^{j\theta}) = j \cdot \omega \cdot A \cdot e^{j\theta} = \omega \cdot A \cdot e^{j(\theta + 90\deg)}$**

- **$\int A \cdot e^{j\theta} \, dt = \frac{1}{j \omega} \cdot A \cdot e^{j\theta} = \frac{1}{\omega} \cdot A \cdot e^{j(\theta - 90\deg)}$**
Lecture 12 Introduction to AC Phasors

Phasor analysis with impedances, For steady-state sinusoidal response ONLY

Sinusoidal AC

![Sinusoidal AC Diagram]

- **T** = Period = repeat time
- **f** = frequency, cycles / second
  \[ f = \frac{1}{T} = \frac{\omega}{2\pi} \]
- **\( \omega \)** = radian frequency, radians/sec
  \[ \omega = 2\pi f \]
- **A** = amplitude

**Phase:**

\[ \phi = \frac{\Delta t}{T} \cdot 360 \text{-deg} \]

**Phasor analysis**

The math is all based on the Euler’s equation

\[
e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)
\]

\[
\cos(\alpha) = \frac{e^{j\alpha} + e^{-j\alpha}}{2}
\]

OR:

\[
\sin(\theta) = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}
\]

If we freeze this at time **t**=0, then we can represent  \( \cos(\omega t + \theta) \) by  \( e^{j\theta} \).

**Voltage:**

\[ v(t) = V_p \cdot \cos(\omega t + \phi) \]

**V(\omega) = V_p e^{j\phi}**

**Current:**

\[ i(t) = I_p \cdot \cos(\omega t + \phi) \]

**I(\omega) = I_p e^{j\phi}**

Phasors are used for adding and subtracting sinusoidal waveforms.

**Ex1.** Add the sinusoidal voltages

\[ v_1(t) = 4.5 \cdot \cos(\omega t - 30 \text{-deg}) \]

and

\[ v_2(t) = 3.2 \cdot \cos(\omega t + 15 \text{-deg}) \]

using phasor notation, draw a phasor diagram of the

three phasors, then convert back to time domain form.

\[ V_1(\omega) = 4.5 \cdot V \cdot e^{j\omega t - 30 \text{-deg}} \]

or:

\[ V_1(\omega) = 4.5 \cdot V \cdot e^{j30\text{-deg}} \]

and

\[ V_2(\omega) = 3.2 \cdot V \cdot e^{j15\text{-deg}} \]

I'm going to drop the (\( \omega \)) notation from the phasor notation, it gets cumbersome, but remember that phasors are in the frequency

domain..

\[ V_1 = 4.5 \cdot V \cdot e^{j30\text{-deg}} \]

or:

\[ V_1 := 4.5 \cdot V \cdot e^{j30\text{-deg}} \]

\[ V_2 = 3.2 \cdot V \cdot e^{j15\text{-deg}} \]

or:

\[ V_2 := 3.2 \cdot V \cdot e^{j15\text{-deg}} \]
Add like vectors, first change to the rectangular form

\[ V_1 = 4.5\,\text{V} / -30^\circ = 4.5\cdot\cos(-30\text{-deg}) = 3.897\cdot\text{V} \]
\[ V_2 = 3.2\,\text{V} / 15^\circ = 3.2\cdot\cos(15\text{-deg}) = 3.091\cdot\text{V} \]

Add real parts:
\[ 3.897 + 3.091 = 6.988 \]
Add imaginary parts:
\[ -2.25 + 0.828 = -1.422 \]

\[ V_3 = 6.988 - 1.422j \cdot \text{V} \]

Change \( V_3 \) back to polar coordinates:
\[ \sqrt{6.988^2 + 1.422^2} = 7.131 \]
\[ \text{atan} \left( \frac{-1.422}{6.988} \right) = -11.502 \text{-deg} \]

OR, in Mathcad notation (you’ll see these in future solutions):
\[ |V_3| = 7.131 \cdot \text{V} \]
\[ \text{arg}(V_3) = -11.5 \cdot \text{deg} \]

Change \( V_3 \) back to the time domain:
\[ v_3(t) = v_1(t) + v_2(t) = 7.13\cdot\cos(\omega t - 11.5\cdot\text{deg}) \cdot \text{V} \]

Ex 2. Two sinusoidal voltages:
\[ v_1(t) = 5\cdot\text{V}\cdot\cos(\omega t + 36.87\cdot\text{deg}) \]
\[ v_2(t) = 3.162\cdot\text{V}\cdot\cos(\omega t - 18.44\cdot\text{deg}) \]

a) using phasor notation, find \( v_3 = v_1 - v_2 \)
\[ V_1 := 5\cdot\text{V}\cdot e^{j(36.87\text{-deg})} \]
\[ V_1 = 4 + 3j \cdot \text{V} \]
\[ V_2 := 3.162\cdot\text{V}\cdot e^{j(-18.44\text{-deg})} \]
\[ V_2 = 3 - j \cdot \text{V} \]

Subtract real parts:
\[ 4\cdot\text{V} - 3\cdot\text{V} = 1\cdot\text{V} \]
Subtract imaginary parts:
\[ 3\cdot\text{V} - (-1)\cdot\text{V} = 4\cdot\text{V} \]

\[ V_3 := V_1 - V_2 \]
\[ V_3 = 1 + 4j \cdot \text{V} \]

Magnitude:
\[ \sqrt{(1\cdot\text{V})^2 + (4\cdot\text{V})^2} = 4.123\cdot\text{V} \]
\[ |V_3| = 4.123\cdot\text{V} \]

Angle:
\[ \text{atan} \left( \frac{4\cdot\text{V}}{1\cdot\text{V}} \right) = 75.96\cdot\text{deg} \]
\[ \text{arg}(V_3) = 75.96\cdot\text{deg} \]

So:
\[ v_3(t) = v_1(t) - v_2(t) = 4.123\cdot\text{V}\cdot\cos(\omega t + 75.96\cdot\text{deg}) \cdot \text{V} \]

What about Capacitors and Inductors?
Capacitors and Inductors in AC circuits cause 90° phase shifts between voltages and currents because they integrate and differentiate. But... integration and differentiation is a piece-of-cake in phasors.
Calculus
\[
\frac{d}{dt}[A e^{j(\omega t + \theta)}] = j\omega A e^{j(\omega t + \theta)} = \omega A e^{j(\omega t + \theta + 90\deg)} = \omega A e^{j(\theta + 90\deg)}
\]
Drop the \(\omega t\) (\(t=0\) to get:
\[
\frac{d}{dt}[A e^{j\theta}] = 0 = \frac{1}{\omega} \cdot A e^{j\theta}
\]

Impedance (like resistance)

**Inductor**
\[
v_L = L \frac{d}{dt} i_L = L \frac{d}{dt} I_p e^{j(\omega t + \theta)} = j\omega L \left[ I_p e^{j(\omega t + \theta)} \right]
\]

in phasor notation ----> \(V_L(\omega) = j\omega L I(\omega)\)

**AC impedance**
\(Z_L = j\omega L\)

**Capacitor**
\[
i_C = C \frac{d}{dt} v_C = C \frac{d}{dt} V_p e^{j(\omega t + \theta)} = j\omega C \left[ V_p e^{j(\omega t + \theta)} \right]
\]

in phasor notation ----> \(I_C(\omega) = j\omega C V(\omega)\)

**Resistor**
\[
v_R = i_R R
\]

You can use impedances just like resistances as long as you deal with the complex arithmetic.
ALL the DC circuit analysis techniques will work with AC.

**series:**
\[
Z_{eq} = Z_1 + Z_2 + Z_3 + \ldots
\]

**Example:**

\(
\omega := 2\pi f = \omega = 3141.6 \text{ rad/sec}
\)
\[
\frac{1}{j\omega C} = -530.516 j \cdot \Omega
\]
\[
Z_{eq} := R + \frac{1}{j\omega C} + j\omega L = 200 \cdot \Omega - 530.5 \cdot j \cdot \Omega + 251.3 \cdot j \cdot \Omega = 200 - 279.2 j \cdot \Omega \text{ rectangular form}
\]
\[
\sqrt{(200)^2 + (279.2)^2} = 343.4 \cdot \Omega
\]
\[
\tan \left( \frac{-279.2}{200} \right) = -54.38 \text{ deg}
\]
\[
Z_{eq} = 343.4 \Omega / -54.4^\circ \text{ polar form}
\]
\[
I := \frac{V}{Z_{eq}} = \frac{12 \cdot V}{343.4 \Omega} = 34.945 \cdot mA \quad \angle 0 - 54.4^\circ = 54.4 \text{ deg}
\]
\[
I = 34.95 mA / -54.4^\circ \quad = I = 20.348 + 28.405 j \cdot mA
\]

**Voltage divider:**
\[
V_\text{Zn} = V_\text{total} \cdot \frac{Z_n}{Z_1 + Z_2 + Z_3 + \ldots}
\]

Note: \(\frac{1}{j} = -j = 1 / -90^\circ\)

\[
\frac{1}{j \omega C} = 12 \cdot V \cdot e^{j0^\circ} \frac{530.516 \cdot e^{j90^\circ} \cdot \Omega}{343.4 \cdot e^{j54.38\deg} \cdot \Omega}
\]
\[
12 \cdot V \cdot \frac{530.516 \cdot \Omega}{343.4 \cdot \Omega} = 18.539 \cdot V \quad \angle 0 - 90 = -35.6 \text{ deg}
\]

\[
V_C = 18.54 V / -35.6^\circ = V_C = 15.069 - 10.795 j \cdot V
\]
parallel:

\[ Z_{eq} = \frac{1}{Z_1 + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots} \]

Example:

\[ f := 500 \text{ Hz} \quad \omega := 2\pi f \quad \omega = 3141.6 \text{ rad sec}^{-1} \]

\[ L := 80 \text{ mH} \quad R := 200 \text{ } \Omega \quad C := 0.6 \mu F \quad \frac{1}{\omega L} = 3.979 \times 10^{-3} \cdot \frac{1}{\Omega} \]

\[ j\omega C = 5 \times 10^{-3} \cdot \frac{1}{\Omega} \]

\[ \omega C = 1.885 \times 10^{-3} \cdot \frac{1}{\Omega} \]

\[ \frac{1}{j\omega L} = -530.516 j \cdot \Omega \]

\[ \frac{1}{j\omega C} = \frac{1}{2.094 \times 10^3 j \cdot \Omega} \]

\[ \omega = \frac{5.4 \times 10^3}{2.93848 \times 10^{-5}} \Omega \]

\[ Z_{eq} = 185.2 22.7^\circ \]

If: \( V := 12 \cdot V e^{j0 \text{-deg}} \quad I := \frac{V}{Z_{eq}} = \frac{12 \cdot V}{185.2 \Omega} = 64.795 \text{mA} \quad \angle 0 - 22.7 = -22.7 \text{ deg} \]

\( I = 60 - 25.127 j \text{ mA} \)

Current divider:

\[ I_{Zn} = I_{total} \frac{1}{Z_n} = \frac{1}{Z_1 + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots} \]

\[ I_{L} := I_{total} \frac{1}{j\omega L} = \frac{1}{R} + \frac{j\omega C}{j\omega L} \frac{1}{j\omega L + j\omega C} = \frac{Z_{eq}}{j\omega L} \]

\[ I_{L} = -47.746 j \text{mA} \]

Duh... \( \frac{V}{j\omega L} = -47.746 j \text{mA} \)

ECE 2210 / 00 Intro to Phasors p4
Ex. 1. Find $V_R$, $V_L$, and $V_C$ in polar phasor form. $f = 2 \cdot \text{kHz}$

- **Find $V_R$, $V_L$, and $V_C$ in polar phasor form.**

  - $V_S = 6.0 \cdot e^{j0}$
  - $R = 500 \Omega$
  - $L = 80 \text{ mH}$
  - $C = 0.4 \text{ µF}$

  $\omega = 2 \cdot \pi \cdot f = 2 \cdot \pi \cdot 2 \cdot \text{kHz} = 1.257 \times 10^4 \text{ rad/sec}$

  - $Z_L = j \cdot \omega \cdot L = 1.005 j \cdot \text{k} \Omega$
  - $Z_C = \frac{1}{j \cdot \omega \cdot C} = -0.199 j \cdot \text{k} \Omega$

  - $Z_{eq} = Z_L + \frac{1}{Z_C} = 500 + 806.366 j \cdot \text{k} \Omega$

  - $Z_{eq} = 500 + 806.366 j \cdot \text{k} \Omega$

  - $\omega^2 = 1.257^2 \times 10^8 \text{ rad}^2/\text{sec}^2$

  - $\tan \left( \frac{806}{500} \right) = 58.187 \text{ deg}$

  - $Z_{eq} = 948.5 \Omega / 58.2^\circ$

  - **Find the current $I = \frac{6 \cdot e^{j0}}{Z_{eq}}$**

    - **Magnitude:** $\frac{6}{948.5 \Omega} = 6.326 \text{mA}$
    - **Angle:** $0^\circ - 58.2^\circ = -58.2^\circ$

    - $I = 6.326 \text{mA} / -58.2^\circ$

  - **Find the magnitude and angle**

    - $V_R = 6.326 \cdot \text{mA} \cdot 500 \Omega = 3.163 \text{V} / -58.2^\circ$
    - $V_L = 6.326 \cdot \text{mA} \cdot 1005 \text{Ω} = 6.358 \text{V} / -90^\circ$
    - $V_C = 6.326 \cdot \text{mA} \cdot (199 \Omega) = -1.259 \text{V} / -90^\circ$

    - **OR:** $V_C = 6.326 \cdot \text{mA} \cdot (199 \Omega) = 1.259 \text{V} / 148.2^\circ$

    - **Find the voltages directly, using a voltage divider.**

      - $V_C = \frac{1}{R + j \cdot \omega \cdot L + \frac{1}{j \cdot \omega \cdot C}} \cdot 6 \cdot V$

      - $= \frac{1}{(1 - \omega^2 \cdot L \cdot C) + j \cdot \omega \cdot R \cdot C} \cdot 6 \cdot V$

      - $= \frac{6 \cdot V}{-4.053 + 2.513 j}$

      - $= \frac{-4.053 - 2.513 j}{(-4.053)^2 + 2.513^2} \cdot 6 \cdot V$

      - $= -24.318 - 15.078 j \cdot \text{V}$

      - $= -24.318 - 15.078 j \cdot \text{V}$

      - $V_C = 1.258 \text{V} / -148.2^\circ$

      - **OR:** $V_C = \frac{1}{R + j \cdot \omega \cdot L + \frac{1}{j \cdot \omega \cdot C}} \cdot 6 \cdot V$

      - $= \frac{1}{(1 - \omega^2 \cdot L \cdot C) + j \cdot \omega \cdot R \cdot C} \cdot 6 \cdot V$

      - $= \frac{6 \cdot V}{-4.053 + 2.513 j}$

      - $= \frac{-4.053 - 2.513 j}{(-4.053)^2 + 2.513^2} \cdot 6 \cdot V$

      - $= -24.318 - 15.078 j \cdot \text{V}$

      - $V_C = 1.258 \text{V} / -148.2^\circ$
Ex. 2. a) Find $Z_{eq}$. \( f = 2.5 \text{ kHz} \) \( \omega := 2 \pi f \) \( \omega = 1.571 \times 10^4 \text{ rad sec} \)

$$Z_{eq} = j \cdot \omega \cdot L_1 + \frac{1}{\frac{1}{j \cdot \omega \cdot C} + \frac{1}{R + j \cdot \omega \cdot L_2}}$$

But it's easier to split the problem up

Left branch

$$Z_L := \frac{1}{j \cdot \omega \cdot C}$$

$$Z_L = -63.662j \cdot \Omega$$

$$\frac{1}{\frac{1}{j \cdot \omega \cdot C} + \frac{1}{R + j \cdot \omega \cdot L_2}} = 0.01571 - j \frac{1}{\Omega}$$

denominator:

$$j \cdot \omega \cdot C + \frac{1}{R + j \cdot \omega \cdot L_2} = 0.01571 - j \left( 3.585 \times 10^{-3} - 2.252 \times 10^{-3}j \right) = 3.585 \times 10^{-3} + 1.346 \times 10^{-2}j$$

rectangular division:

$$\frac{1}{\left( 3.585 \times 10^{-3} - 1.346 \times 10^{-2}j \right)^2 + \left( 3.585 \times 10^{-3} + 1.346 \times 10^{-2}j \right)^2} = 1.94 \times 10^{-4}$$

$$\frac{1}{\left( 3.585 \times 10^{-3} - 1.346 \times 10^{-2}j \right)^2} = 18.479 - 69.381j$$

$$j \cdot \omega \cdot L_1 = 31.416j \cdot \Omega$$

$$31.416j + (18.479 - 69.381j) = 18.479 - 37.965j \cdot \Omega$$

convert to polar (if needed):

$$\sqrt{18.48^2 + 37.97^2} = 42.228$$

$$\tan \left( \frac{-37.97}{18.48} \right) = -64.048 \text{ deg}$$

$$Z_{eq} = 42.23 \Omega / -64.05^\circ$$

Another Way

Sometimes you might simplify a little before putting in numbers.

$$Z_{eq} := j \cdot \omega \cdot L_1 + \frac{1}{\frac{1}{R + j \cdot \omega \cdot L_2} + \frac{1}{j \cdot \omega \cdot C}}$$

$$= j \cdot \omega \cdot L_1 + \frac{1}{\frac{R + j \cdot \omega \cdot L_2}{1 + j \cdot \omega \cdot C \cdot \left( R + j \cdot \omega \cdot L_2 \right)}}$$

$$= j \cdot \omega \cdot L_1 + \frac{R + j \cdot \omega \cdot L_2}{1 - \frac{j \cdot \omega \cdot C \cdot \left( R + j \cdot \omega \cdot L_2 \right)}{j \cdot \omega \cdot C \cdot R}}$$

$$Z_{eq} = 31.416j \cdot \Omega + \frac{(200 + 125.664j) \cdot \Omega}{-0.974 + 3.142j} \left( \frac{-0.974 - 3.142j}{-0.974 - 3.142j} \right) = 31.416j \cdot \Omega + \frac{200 + 125.664j}{0.974^2 + 3.142^2}$$

$$= 31.416j \cdot \Omega + \frac{200 + 125.664j - 125.664(-3.142j) + (125.664(-0.974) - 200(-3.142)j)}{0.974^2 + 3.142^2}$$

$$= 31.416j \cdot \Omega + \frac{200.036288 - 750.796736j}{10.82084} = 31.416j \cdot \Omega + \frac{200.036288 - 750.796736j}{10.82084} = 31.416j \cdot \Omega + 18.486 \Omega - 69.384j \cdot \Omega = 18.486 - 37.968j \cdot \Omega$$

$$\sqrt{18.49^2 + 37.97^2} = 42.233$$

$$\tan \left( \frac{-37.97}{18.49} \right) = -64.036 \text{ deg}$$

$$Z_{eq} = 42.23 \Omega / -64.04^\circ$$

a little roundoff difference
b) \( V_{\text{in}} := 12\cdot e^{j20\text{-deg}} \) Find \( I_{L1}, V_C \)

\[
I_{L1} := \frac{V_{\text{in}}}{Z_{\text{eq}}} = \frac{12V}{42.23\Omega} = 284.16\text{mA} \quad 20\text{-deg} - (-64.04)\text{-deg} = 84.04\text{-deg}
\]

\[
I_{L1} = 284\text{mA} / 84.04^\circ
\]

\[
V_C := I_{L1} \cdot (18.479 - 69.381j)\Omega
\]

\[
284\text{-mA} \cdot \sqrt{18.479^2 + 69.381^2} = 20.391\cdot V
\]

\[
84.04\text{-deg} + \arctan\left(-\frac{69.381}{18.479}\right) = 8.954\text{-deg}
\]

You could then use another voltage divider to find \( V_R \) or \( V_{L2} \).

Another Way

To find \( V_C \) directly:

\[
V_C := \frac{1}{R + j\omega C + j\omega L_1 + \frac{1}{R + j\omega L_2}} \cdot V_{\text{in}} \quad \rightarrow \text{math} \rightarrow \quad V_C = 20.153 + 3.178j \cdot V
\]

Same but for a little roundoff difference.

c) Let's find \( I_{L2} \).

\[
Z_r = 200 + 125.664j \cdot \Omega \quad \sqrt{200^2 + 125.664^2} = 236.202 \quad \arctan\left(\frac{125.664}{200}\right) = 32.142\text{-deg}
\]

\[
I_{L2} := \frac{V_C}{Z_r} = \frac{20.4\cdot V\cdot e^{j8.95^\circ}}{236.202\cdot \Omega\cdot e^{j32.142^\circ}} = \frac{20.4\cdot V}{236.202\cdot \Omega} / (8.95 - 32.142^\circ) = 86.4\text{mA} / -23.19^\circ
\]

Another Way

Directly by Current divider:

\[
I_{L2} := \frac{1}{R + j\omega L_2} - I_{L1} = \frac{1}{j\omega C + \frac{1}{R + j\omega L_2}} - I_{L1} = \frac{I_{L1}}{1 - \omega^2C\cdot L_2 + j\omega C\cdot R}
\]

\[
\text{real part is negative}
\]

\[
I_{L2} = \frac{284\text{-mA}\cdot e^{j84.04^\circ}}{3.289\cdot e^{-107.224^\circ}} = \frac{284\text{-mA}}{3.289} / 84.04 - 107.224^\circ = 86.4\text{mA} / -23.18^\circ
\]

d) How about \( I_C \)?

\[
I_C := \frac{V_C}{j\omega C} = V_C \cdot j\omega C = 20.4V / 8.95^\circ \quad 0.015708 / 90^\circ \quad \frac{1}{\Omega} = 320\text{mA} / 98.95^\circ
\]

Another Way

Could also be found directly by current divider:

\[
I_C := \frac{j\omega C}{j\omega C + \frac{1}{R + j\omega L_2}} - I_{L1} = 320\text{mA} / 98.95^\circ
\]

Something Weird

\( I_C \) is greater than the input current (\( I_{L1} \)). What's going on?

The angle between \( I_C \) & \( I_{L2} \) is big enough that they somewhat cancel each other out (partially resonate).

Check Kirchoff's Current Law: \( I_C + I_{L2} = 29.485 + 282.569j \cdot \text{mA} = I_{L1} = 29.485 + 282.569j \cdot \text{mA} \) yes

ECE 2210 / 00 Phasor Examples p3
Ex 3. a) Find $Z_2$.

\[ I := 25\text{m}A \cdot e^{j10\text{deg}} \]
\[ V_\text{in} := 10\text{V} \cdot e^{-j20\text{deg}} \]
\[ Z_T := \frac{V_\text{in}}{I} = \frac{10\text{V}}{25\text{mA}} \cdot \frac{1}{\angle 20 - 10^\circ} = 400\Omega \angle 30^\circ \]
\[ Z_T = 346.41 - 200j \cdot \Omega \]
\[ Z_2 = Z_T - R - Z_1 = (346.41 - 200j) \cdot \Omega - 50\Omega - (120 - 60j) \cdot \Omega = 176.41 - 140j \cdot \Omega \]

b) Circle 1: i) The source current leads the source voltage
\[
\text{answer, because } 10^\circ > -20^\circ. \]
ii) The source voltage leads the source current

Ex 4. a) Find $V_\text{in}$ in polar form.

\[ I_Z := 100\text{mA} \]
\[ Z := (80 - 60j) \cdot \Omega \]
\[ \omega := 1000\text{rad/sec} \]
\[ V_\text{in} := I_Z \cdot Z \]
\[ V_\text{in} = 8 - 6j \cdot V \]
\[ \sqrt{8^2 + 6^2} = 10 \quad \text{atan} \left( \frac{6}{8} \right) = -36.87 \cdot \text{deg} \]
\[ V_\text{in} = 10\text{V} \angle -36.9^\circ \]

b) Find $I_T$ in polar form.

\[ I_R := \frac{V_\text{in}}{R} = \frac{10\text{V}}{50\Omega} \cdot \frac{1}{\angle -36.9^\circ} = \frac{10\text{V}}{50\Omega} \cdot \cos(-36.9\text{deg}) + j\frac{10\text{V}}{50\Omega} \cdot \sin(-36.9\text{deg}) = 160 - 120i \cdot \text{mA} \]
\[ I_T := I_R + I_Z = (160 - 120j) \cdot \text{mA} + 100 \cdot \text{mA} = 260 - 120j \cdot \text{mA} \]
\[ \sqrt{260^2 + 120^2} = 286.356 \quad \text{atan} \left( \frac{120}{260} \right) = -24.78 \cdot \text{deg} \quad I_T = 286\text{mA} \angle -24.8^\circ \]

c) Circle 1: i) The source current leads the source voltage
\[
\text{answer i), } -24.8^\circ > -36.9^\circ. \]
ii) The source voltage leads the source current

d) The impedance $Z$ (above) is made of two components in series. What are they and what are their values?

\[ Z = 80 - 60j \cdot \Omega \]

Must have a resistor because there is a real part.

\[ R := \text{Re}(Z) \\ R = 80 \cdot \Omega \]

Must have a capacitor because the imaginary part is negative.

\[ \text{Im}(Z) = -60 \cdot \Omega = \frac{-1}{\omega \cdot C} \quad C := \frac{-1}{\omega \cdot \text{Im}(Z)} \quad C = 16.667 \cdot \mu\text{F} \]
Ex 5. The impedance $Z = 80 - 60j \ \Omega$ is made of two components in parallel. What are they and what are their values?

Must have a resistor because there is a real part.

Must have a capacitor because the imaginary part is negative.

$$Z = \frac{1}{R + j\omega C}$$

$$\frac{1}{Z} = 0.008 + 0.006i \ \Omega^{-1} = \frac{1}{R + j\omega C}$$

$$\frac{1}{R} = 0.008 \ \Omega^{-1} \quad R := \frac{1}{0.008 \ \Omega^{-1}} \quad R = 125 \ \Omega$$

$$\omega C = 0.006 \ \Omega^{-1} \quad C := \frac{0.006 \ \Omega^{-1}}{\omega} \quad C = 6 \ \mu F$$

Positive imaginary parts would require inductors

Ex 6. a) Find $I_1$

$$\omega := 20000 \ \text{rad/sec}$$

$$V_{\text{in}} := 20 \cdot V \cdot e^{i 30\text{-deg}}$$

$$I_1 := \frac{V_{\text{in}}}{R} = \frac{20 \cdot V}{250 \ \Omega} \cdot e^{i 30\text{-deg}} = 80 \cdot mA \cdot e^{i 30\text{-deg}}$$

polar division

b) Circle 1:

i) $V_{\text{in}}$ leads $I_2$

ii) $V_{\text{in}}$ lags $I_2$

Why? Show numbers: $30 > 20$

< _____

c) Find $Z_2$ in polar form

Convert $V_{\text{in}}$ to rectangular coordinates

$$20 \cdot V \cdot \cos(30\text{-deg}) = 17.321 \cdot V$$

$$20 \cdot V \cdot \sin(30\text{-deg}) = 10 \cdot V$$

pol to rect

$$V_{\text{in}} = 17.321 + 10j \cdot V$$

$$V_{Z2} := V_{\text{in}} - V_{Z1}$$

$$V_{Z2} = 9.321 + 15j \cdot V$$

subtract

rect to pol

$$\sqrt{9.321^2 + 15^2} = |V_{Z2}| = 17.66 \cdot V$$

$$\tan \left( \frac{15}{9.321} \right) = \arg(V_{Z2}) = 58.145 \ \text{deg}$$

$$\text{div} \ Z_2 := \frac{V_{Z2}}{I_2} = \frac{17.66 \cdot V}{20 \cdot mA} = 883 \ \Omega$$

$$58.145 \ \text{deg} - 20 \ \text{deg} = 38.145 \ \text{deg}$$

$$Z_2 = 883 / 38.15^\circ \ \Omega$$

$$Z_2 = 694.436 + 545.379j \ \Omega$$
Ex 7. You need to design a circuit in which the "output" voltage leads the input voltage ($v_S(t)$) by $40^\circ$ of phase.

a) What should go in the box: R, L, C?

$$V_o = \frac{Z_{\text{box}}}{R + Z_{\text{box}}} \cdot V_S$$

angle of $\frac{Z_{\text{box}}}{R + Z_{\text{box}}}$ is $40^\circ$.

This can only happen if the angle of $Z_{\text{box}}$ is positive, so $Z_{\text{box}}$ is an inductor

b) Find its value. $V_o = \frac{\frac{j \cdot \omega \cdot L}{R + j \cdot \omega \cdot L}}{R + \frac{j \cdot \omega \cdot L}{R}} \cdot V_S$ angle $\frac{j \cdot \omega \cdot L}{R}$ is $90^\circ - \frac{\omega \cdot L}{R}$ = $40^\circ$.

So: $\tan\left(\frac{\omega \cdot L}{R}\right) = 50^\circ$, $\frac{\omega \cdot L}{R} = \tan(50\text{-deg}) = 1.192$, $L = \frac{R \cdot 1.192}{\omega} = 75.9 \text{ mH}$

c) Repeat if the "output" voltage should lag the input voltage ($v_S(t)$) by $20^\circ$ of phase.

angle of $\frac{Z_{\text{box}}}{R + Z_{\text{box}}}$ is $-20^\circ$. This can only happen if the angle of $Z_{\text{box}}$ is negative, so $Z_{\text{box}}$ is a capacitor

$$V_o = \frac{\frac{1}{R + \frac{1}{j \cdot \omega \cdot C}}}{R + \frac{1}{j \cdot \omega \cdot C}} \cdot V_S$$

angle $\frac{1}{R + \frac{1}{j \cdot \omega \cdot C}}$ is $-90^\circ - \frac{1}{\cdot \omega \cdot C \cdot R}$ = $-90^\circ - \frac{1}{\cdot \omega \cdot C \cdot R}$

$\tan(-70\text{-deg}) = -2.747$, $C = \frac{1}{\omega \cdot R \cdot 2.747} = 0.145 \mu F$

Ex 8. Find $V_O$ in the circuit shown. Express it as a magnitude and phase angle (polar).

$$V_o : \frac{V_S}{Z_1}$$

$$V_O : \frac{Z_2}{Z_1 + Z_2} \cdot V_S$$

Simple voltage divider

$|Z_2| \cdot \cos(-60\text{-deg}) = 40\cdot \Omega$, $|Z_2| \cdot \sin(-60\text{-deg}) = -69.282 \cdot \Omega$, $Z_2 = 40 - 69.282j \cdot \Omega$

$Z_1 + Z_2 = 25 \cdot \Omega + 35j \cdot \Omega + 40 \cdot \Omega - 69.282j \cdot \Omega = 65 - 34.282j \cdot \Omega = 73.486 \cdot \Omega \cdot e^{j27.81^\circ}$

$$V_O : \frac{Z_2}{Z_1 + Z_2} \cdot V_S = \frac{80 \cdot \Omega \cdot e^{j60^\circ}}{73.486 \cdot \Omega \cdot e^{j27.81^\circ}} \cdot (6 \cdot \Omega \cdot e^{j18^\circ}) = \frac{80 \cdot \Omega}{73.486 \cdot \Omega} \cdot 6 \cdot \Omega \cdot e^{j(60^\circ - (27.81 + 18)^\circ)} = 6.53 \cdot \Omega \cdot e^{j14.2^\circ}$