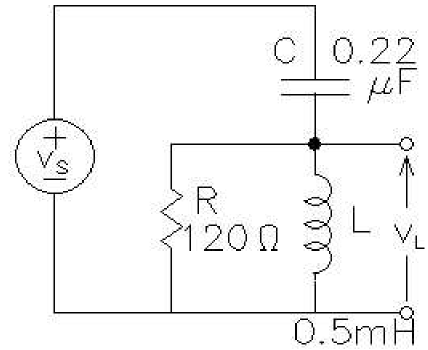
A. Stolp
10/30/06
2/19/10

Ex. 1 For the circuit shown:



a) Find the transfer function v_L.

$$\mathbf{V_L}(s) = \frac{\dfrac{1}{\dfrac{1}{L\cdot s}+\dfrac{1}{R}}}{\dfrac{1}{\dfrac{1}{L\cdot s}+\dfrac{1}{R}}+\dfrac{1}{C\cdot s}}\cdot \mathbf{V_S}(s)$$

$$= \frac{1}{1+\dfrac{1}{C\cdot s}\cdot\left(\dfrac{1}{L\cdot s}+\dfrac{1}{R}\right)}\cdot\mathbf{V_S}(s) = \frac{1}{1+\dfrac{1}{C\cdot s}\cdot\dfrac{1}{L\cdot s}+\dfrac{1}{C\cdot s}\cdot\dfrac{1}{R}}\cdot\mathbf{V_S}(s) = \frac{s^2}{s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{L\cdot C}}\cdot\mathbf{V_S}(s)$$

$$\mathbf{H}(s) = \frac{\mathbf{V_L}(s)}{\mathbf{V_S}(s)} = \frac{s^2}{s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{L\cdot C}}$$

$$= \frac{s^2}{s^2+\dfrac{3.788\cdot 10^4}{sec}\cdot s+\dfrac{9.091\cdot 10^9}{sec^2}}$$

$R := 120\cdot\Omega$ $C := 0.22\cdot\mu F$ $L := 0.5\cdot mH$

$\frac{1}{C\cdot R} = 3.788\cdot 10^4 \cdot \frac{1}{sec}$ $\frac{1}{L\cdot C} = 9.091\cdot 10^9 \cdot \frac{1}{sec^2}$

b) Find the characteristic equation for this circuit.

$$0 = s^2+\frac{1}{C\cdot R}\cdot s+\frac{1}{L\cdot C} = s^2+\frac{3.788\cdot 10^4}{sec}\cdot s+\frac{9.091\cdot 10^9}{sec^2}$$

Just the denominator set to zero. The solutions of the characteristic equation are the "poles" of the transfer function.

c) Find the differential equation for v_L.

Cross-multiply the transfer function

$$s^2\cdot\mathbf{V_S}(s) = \left(s^2+\frac{1}{C\cdot R}\cdot s+\frac{1}{L\cdot C}\right)\cdot\mathbf{V_L}(s)$$

$$s^2\cdot\mathbf{V_S}(s) = s^2\cdot\mathbf{V_L}(s)+\frac{1}{C\cdot R}\cdot s\cdot\mathbf{V_L}(s)+\frac{1}{L\cdot C}\cdot\mathbf{V_L}(s)$$

$$\frac{d^2}{dt^2}v_S(t) = \frac{d^2}{dt^2}v_L(t)+\frac{1}{C\cdot R}\cdot\frac{d}{dt}v_L(t)+\frac{1}{L\cdot C}\cdot v_L(t)$$

$$\frac{d^2}{dt^2}v_S(t) = \frac{d^2}{dt^2}v_L(t)+\frac{3.788\cdot 10^4}{sec}\cdot\frac{d}{dt}v_L(t)+\frac{9.091\cdot 10^9}{sec^2}\cdot v_L(t)$$

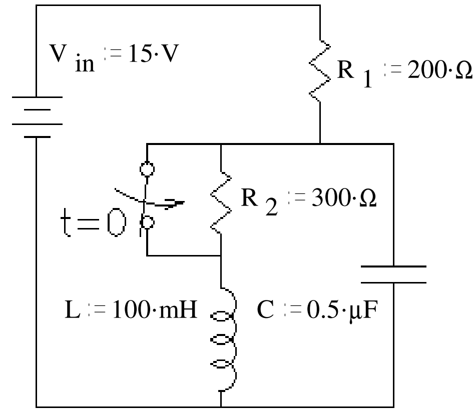
d) What are the solutions to the characteristic equation?

$$s_1 = \frac{-3.788\cdot 10^4}{2}+\frac{1}{2}\cdot\sqrt{\left(3.788\cdot 10^4\right)^2-4\cdot\left(9.091\cdot 10^9\right)} = -1.894\cdot 10^4 + 9.345\cdot 10^4 j$$

$$s_2 = \frac{-3.788\cdot 10^4}{2}-\frac{1}{2}\cdot\sqrt{\left(3.788\cdot 10^4\right)^2-4\cdot\left(9.091\cdot 10^9\right)} = -1.894\cdot 10^4 - 9.345\cdot 10^4 j$$

e) What type of response do you expect from this circuit? The solutions to the characteristic equation are complex so the response will be **underdamped**.

Ex. 2 Analysis of the circuit shown yields the characteristic equation below. The switch has been in the open position for a long time and is closed (as shown) at time $t = 0$. Find the initial and final conditions and write the full expression for $i_L(t)$, including all the constants that you find.



$$s^2 + \left(\frac{1}{C \cdot R_1}\right) \cdot s + \left(\frac{1}{L \cdot C}\right) = 0$$

$$\left(\frac{1}{C \cdot R_1}\right) = 1 \cdot 10^4 \cdot \frac{1}{sec} \qquad \left(\frac{1}{L \cdot C}\right) = 2 \cdot 10^7 \cdot \frac{1}{sec^2}$$

$$s^2 + 10000 \cdot \frac{1}{sec} \cdot s + 2 \cdot 10^7 \cdot \frac{1}{sec^2} = 0$$

$$s_1 := \left[\frac{-10000}{2} + \frac{1}{2} \cdot \sqrt{(10000)^2 - 4 \cdot \left(2 \cdot 10^7\right)}\right] \cdot sec^{-1} \qquad s_2 := \left[\frac{-10000}{2} - \frac{1}{2} \cdot \sqrt{(10000)^2 - 4 \cdot \left(2 \cdot 10^7\right)}\right] \cdot sec^{-1}$$

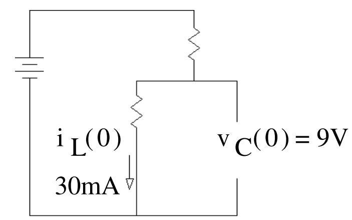
$$s_1 = -2764 \cdot sec^{-1} \qquad s_2 = -7236 \cdot sec^{-1}$$

s_1 and s_2 are both real and distinct, overdamped

Find the initial conditions:

Before the switch closed, the inductor current was: $\frac{15 \cdot V}{R_1 + R_2} = 30 \cdot mA \; = \; i_L(0)$

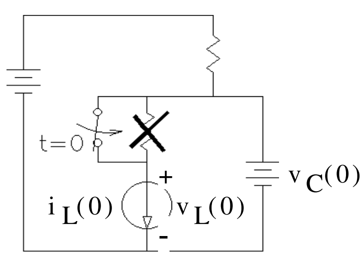
Before the switch closed, the capacitor voltage was: $\frac{R_2}{R_1 + R_2} \cdot (15 \cdot V) = 9 \cdot V \; = \; v_C(0)$



When the switch is closed, the inductor is suddenly in parallel with the capacitor, and:

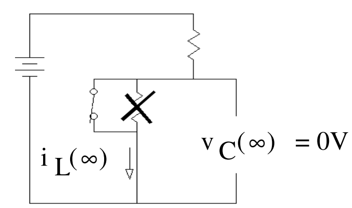
$$v_L(0) = v_C(0)$$

$$\frac{d}{dt} i_L(0) = \frac{1}{L} \cdot v_L(0) = \frac{1}{L} \cdot 9 \cdot V = 90 \cdot \frac{A}{sec}$$



Find the final condition:

$$i_L(\infty) = \frac{15 \cdot V}{R_1} = 75 \cdot mA$$



General solution for the overdamped condition: $i_L(t) = i_L(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$

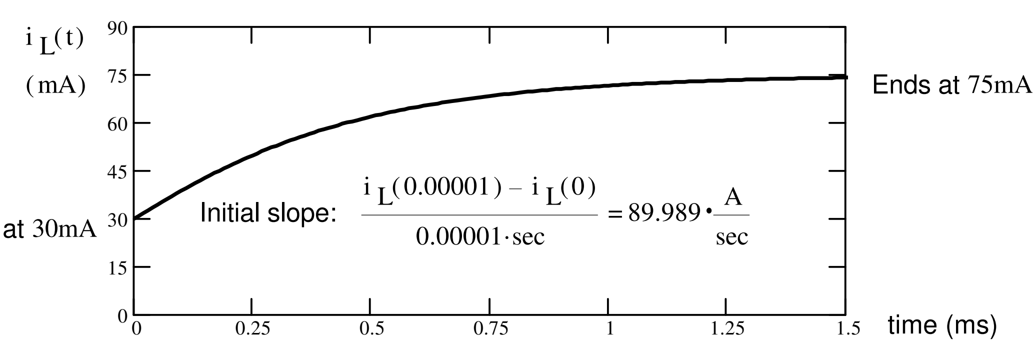
Initial conditions: $i_L(0) = \frac{15 \cdot V}{R_1 + R_2} = i_L(\infty) + B + D$, so $B = i_L(0) - i_L(\infty) - D = 30 \cdot mA - 75 \cdot mA - D$

$$= -45 \cdot mA - D$$

$$\frac{d}{dt} i_L(0) = 90 \cdot \frac{A}{sec} = s_1 \cdot B + s_2 \cdot D = s_1 \cdot (-45 \cdot mA - D) + s_2 \cdot D = s_1 \cdot (-45 \cdot mA) - s_1 \cdot D + s_2 \cdot D$$

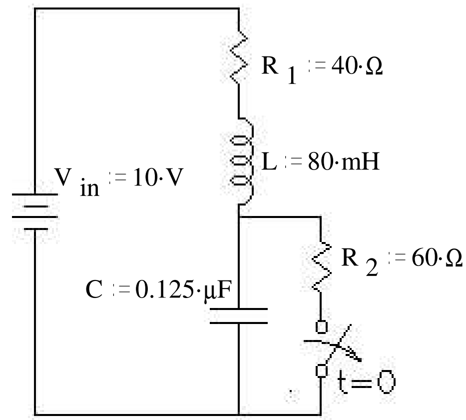
solve for D & B: $D := \frac{90 \cdot \frac{A}{sec} - s_1 \cdot (-45 \cdot mA)}{-s_1 + s_2}$ $\qquad D = 7.69 \cdot mA \qquad B := -45 \cdot mA - D \qquad B = -52.7 \cdot mA$

Plug numbers back in: $i_L(t) := 75 \cdot mA - 52.7 \cdot mA \cdot e^{-2764t} + 7.69 \cdot mA \cdot e^{-7236t}$



Ex. 3

Analysis of the circuit shown yields the characteristic equation and s values below. The switch has been in the closed position for a long time and is opened (as shown) at time $t = 0$. Find the initial and final conditions and write the full expression for $v_C(t)$, including all the constants.



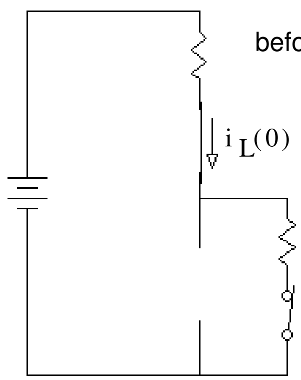
$$0 = s^2 + \frac{R_1}{L}\cdot s + \frac{1}{L\cdot C}$$

$$s_1 := \left(-250 + 10^4\cdot j\right)\cdot\frac{1}{\text{sec}} \quad , \qquad s_2 := \left(-250 - 10^4\cdot j\right)\cdot\frac{1}{\text{sec}}$$

Solution: $\alpha := -250\cdot\frac{1}{\text{sec}}$ $\qquad \omega := 10000\cdot\frac{\text{rad}}{\text{sec}}$

Initial conditions:

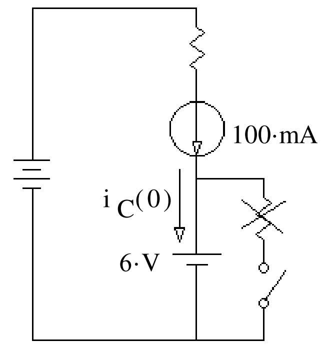
before switch opens



$$i_L(0) = \frac{V_{in}}{R_1 + R_2} = 100\cdot\text{mA}$$

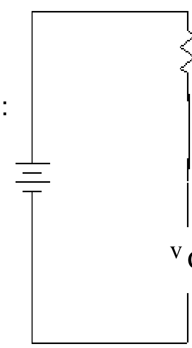
$$v_C(0) = V_{in}\cdot\frac{R_2}{R_1 + R_2} = 6\cdot\text{V}$$

just after the switch opens



$$\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C} = \frac{100\cdot\text{mA}}{C} = 8\cdot 10^5\cdot\frac{\text{V}}{\text{sec}}$$

Find final condition:



$$v_C(\infty) = V_{in} = 10\cdot\text{V}$$

Find constants: $v_C(0) = v_C(\infty) + B$ $\qquad B = v_C(0) - v_C(\infty)$ $\qquad B := 6\cdot\text{V} - 10\cdot\text{V}$ $\qquad B = -4\cdot\text{V}$

$$\frac{d}{dt}v_C(0) = \alpha\cdot B + D\cdot\omega \qquad D := \frac{8\cdot 10^5\cdot\frac{\text{V}}{\text{sec}} - \alpha\cdot B}{\omega} \qquad D = 79.9\cdot\text{V}$$

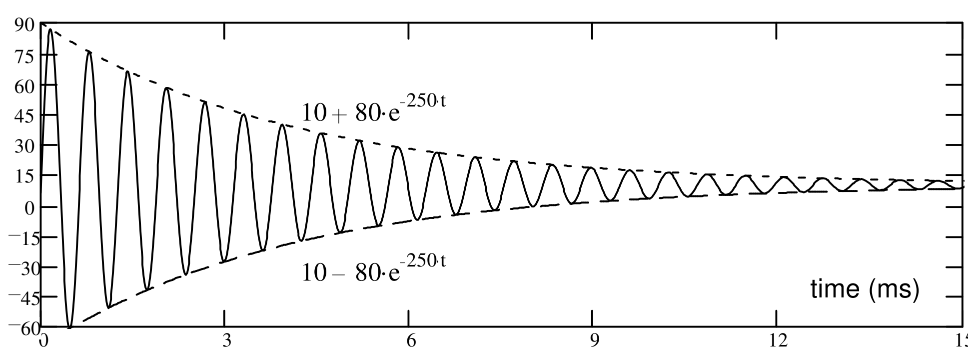
Write the full expression for $v_C(t)$, including all the constants that you find.

$$v_C(t) = e^{\alpha t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t)) + v_C(\infty)$$

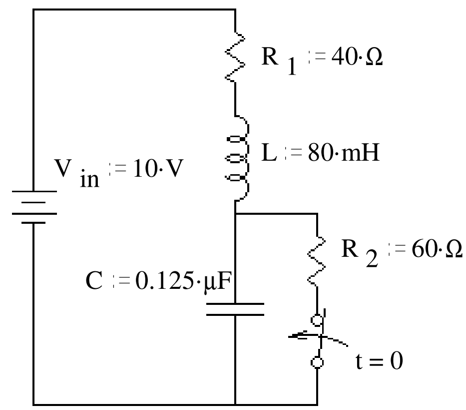
$$v_C(t) := e^{-250t}\cdot\left(-4\cdot\text{V}\cdot\cos\left(10^4\cdot t\right) + 79.9\cdot\text{V}\cdot\sin\left(10^4\cdot t\right)\right) + 10\cdot\text{V}$$

$$\sqrt{D^2 + B^2} = 80\cdot\text{V}$$

$v_C(t)$ (volts)



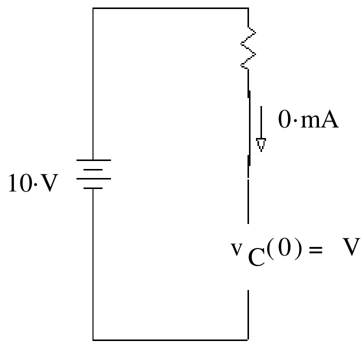
Ex. 4 Ex.3 Backwards, switch closes at $t = 0$



Characteristic eq.: $0 = s^2 + \left(\frac{1}{C\cdot R_2} + \frac{R_1}{L}\right)\cdot s + \left(1 + \frac{R_1}{R_2}\right)\cdot\frac{1}{L\cdot C}$

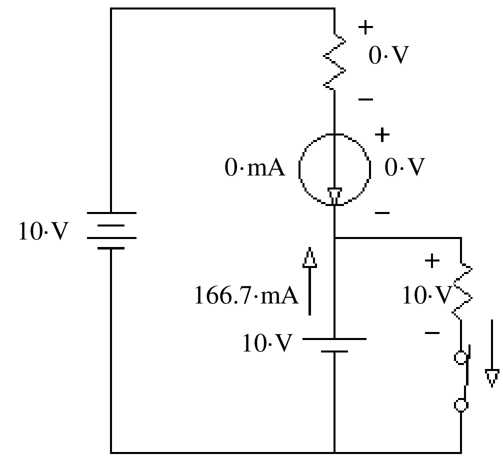
$s_1 := -1.257\cdot 10^3\cdot\frac{1}{sec}$ $\quad s_2 := -1.326\cdot 10^5\cdot\frac{1}{sec}$

Initial conditions, same as Ex.3 final:



$v_C(0) = V_{in} = 10\cdot V$

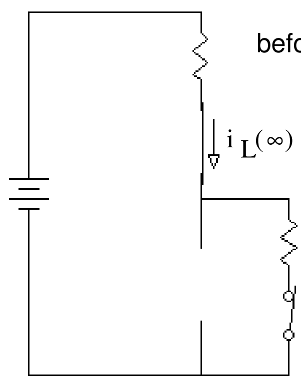
just after the switch opens



$\frac{10\cdot V}{R_2} = 166.7\cdot mA$

$\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C} = \frac{-166.7\cdot mA}{C} = -1.334\cdot 10^6\cdot\frac{V}{sec}$

Find final condition:



before switch opens

$i_L(\infty) = \frac{V_{in}}{R_1 + R_2} = 100\cdot mA$

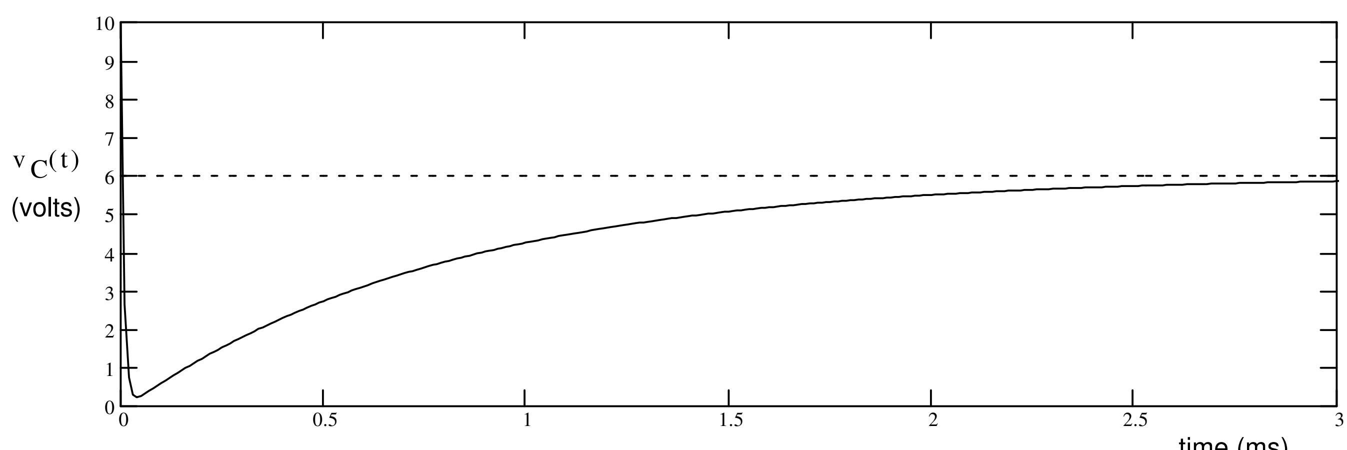
$v_C(\infty) = V_{in}\cdot\frac{R_2}{R_1 + R_2} = 6\cdot V$

Find constants: $v_C(0) = v_C(\infty) + B + D$, so $B = v_C(0) - v_C(\infty) - D = 10\cdot V - 6\cdot V - D = 4\cdot V - D$

$\frac{d}{dt}v_C(0) = -1.334\cdot 10^6\cdot\frac{V}{sec} = s_1\cdot B + s_2\cdot D = s_1\cdot(4\cdot V - D) + s_2\cdot D = s_1\cdot(4\cdot V) - s_1\cdot D + s_2\cdot D$

$D := \frac{-1.334\cdot 10^6\cdot\frac{V}{sec} - s_1\cdot(4\cdot V)}{-s_1 + s_2}$ $\quad D = 10.12\cdot V$ $\quad B := 4\cdot V - D$ $\quad B = -6.12\cdot V$

$v_C(t) := 6\cdot V - 6.12\cdot V\cdot e^{-1257\,t} + 10.12\cdot V\cdot e^{-132600\cdot t}$



Ex. 5 Analysis of a circuit (not pictured) yields the characteristic equation below.

$0 = s^2 + 400 \cdot s + 400000$ $\qquad R := 80 \cdot \Omega \qquad L := 20 \cdot mH \qquad C := 2 \cdot \mu F$

Further analysis yields the followiing initial and final conditions:

$i_L(0) = 120 \cdot mA \qquad v_L(0) = -3 \cdot V \qquad v_C(0) = 7 \cdot V \qquad i_C(0) = -80 \cdot mA$

$i_L(\infty) = 800 \cdot mA \qquad v_L(\infty) = 0 \cdot V \qquad v_C(\infty) = 12 \cdot V \qquad i_C(\infty) = 0 \cdot mA$

Write the full expression for $i_L(t)$, including all the constants that you find. $\qquad i_L(t) = ?$

Solution:

$$\frac{400}{2} = 200 \qquad \frac{\sqrt{400^2 - 4 \cdot 400000}}{2} = 600j$$

$s_1 := (-200 + 600 \cdot j) \cdot \frac{1}{sec}$ and $s_2 := (-200 - 600 \cdot j) \cdot \frac{1}{sec}$

$\alpha := Re(s_1) \qquad \alpha = -200 \cdot sec^{-1} \qquad \omega := Im(s_1) \qquad \omega = 600 \cdot sec^{-1}$

Initial slope: $\frac{d}{dt} i_L(0) = \frac{v_L(0)}{L} = \frac{-3 \cdot V}{L} = -150 \cdot \frac{A}{sec}$

General solution for the underdamped condition: $i_L(t) = i_L(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$

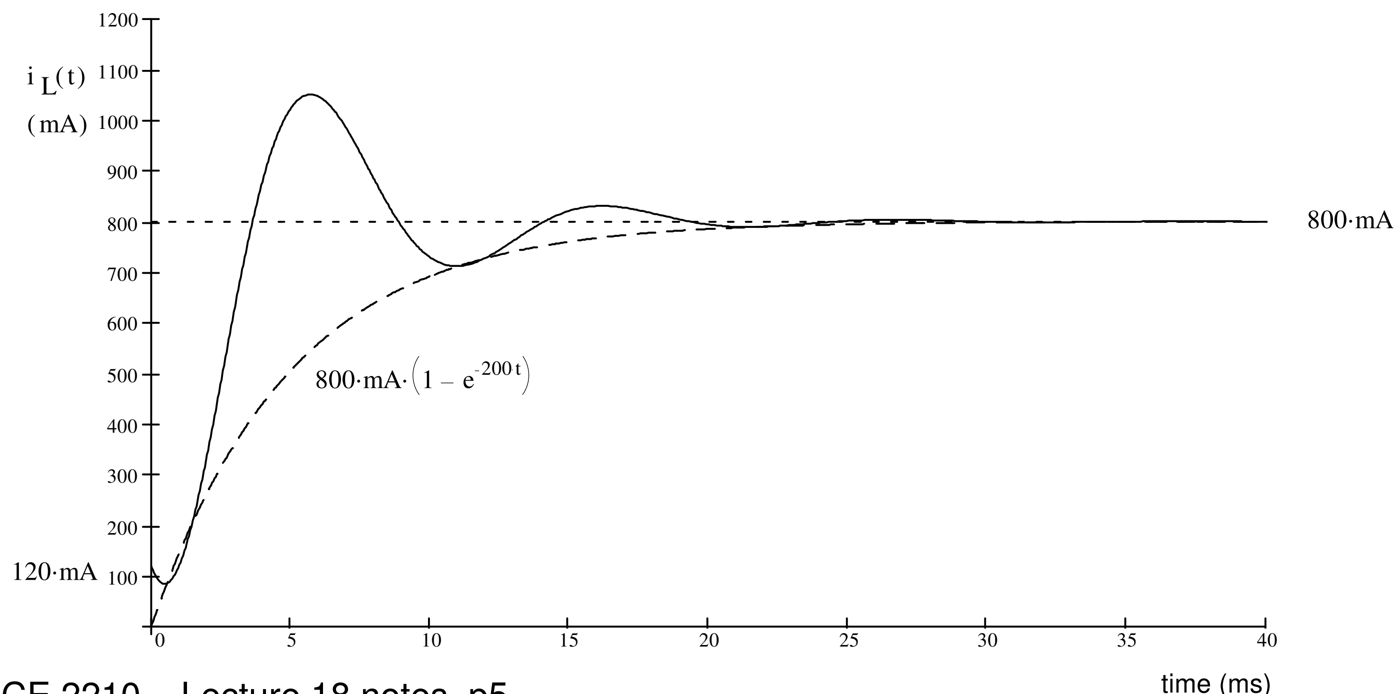
Find constants: $i_L(0) = i_L(\infty) + B \qquad B = i_L(0) - i_L(\infty) \qquad B := 120 \cdot mA - 800 \cdot mA$

$B = -680 \cdot mA$

$\frac{d}{dt} i_L(0) = \alpha \cdot B + D \cdot \omega \qquad D := \frac{-150 \cdot \frac{A}{sec} - \alpha \cdot B}{\omega} \qquad D = -476.667 \cdot mA$

Write the full expression for $i_L(t)$, including all the constants that you find.

$i_L(t) := 800 \cdot mA + e^{-200t} \cdot (-680 \cdot mA \cdot \cos(600 \cdot t) - 477 \cdot mA \cdot \sin(600 \cdot t))$



Ex. 6

Analysis of a circuit (not pictured) yields the characteristic equation below.

$$0 = s^2 + 800 \cdot s + 160000 \qquad R := 60 \cdot \Omega \qquad L := 350 \cdot mH \qquad C := 20 \cdot \mu F \qquad V_{in} := 12 \cdot V$$

Further analysis yields the following initial and final conditions:

$$i_L(0) = 30 \cdot mA \qquad v_L(0) = -7 \cdot V \qquad v_C(0) = 5 \cdot V \qquad i_C(0) = 70 \cdot mA$$

$$i_L(\infty) = 90 \cdot mA \qquad v_L(\infty) = 0 \cdot V \qquad v_C(\infty) = 12 \cdot V \qquad i_C(\infty) = 0 \cdot mA$$

Write the full expression for $i_L(t)$, including all the constants that you find. $i_L(t) = ?$

Include **units** in your answer

Solution:

$$\frac{-800 + \sqrt{800^2 - 4 \cdot 160000}}{2} = -400 \qquad s_1 := -400 \cdot \frac{1}{sec} \qquad s_2 := -400 \cdot \frac{1}{sec}$$

s_1 and s_2 are the same, **critically damped**

Initial slope: $\frac{d}{dt} i_L(0) = \frac{v_L(0)}{L} = \frac{-7 \cdot V}{L} = -20 \cdot \frac{A}{sec}$

General solution for the critically damped condition: $i_L(t) = i_L(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot t \cdot e^{s_2 \cdot t}$

Find constants: $i_L(0) = i_L(\infty) + B \qquad B = i_L(0) - i_L(\infty) \qquad B := 30 \cdot mA - 90 \cdot mA$

$B = -60 \cdot mA$

$\frac{d}{dt} i_L(0) = B \cdot s + D \qquad D := -20 \cdot \frac{A}{sec} - B \cdot s_1 \qquad D = -44 \cdot \frac{A}{sec}$

Write the full expression for $i_L(t)$, including all the constants that you find.

$$i_L(t) := 90 \cdot mA - 60 \cdot mA \cdot e^{-\frac{400}{sec} \cdot t} - 44 \cdot \frac{A}{sec} \cdot t \cdot e^{-\frac{400}{sec} \cdot t}$$