ECE 2210 Lecture 18 notes Second order Transient examples

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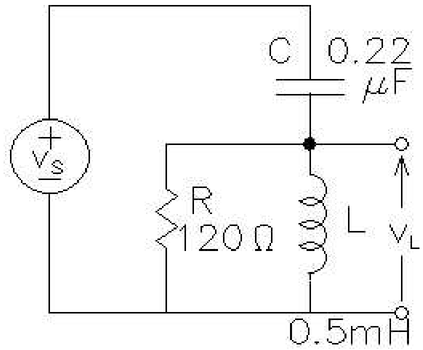
10/30/06
2/27/07

Ex 1. For the circuit shown:



a) Find the transfer function v_L.

$$\mathbf{V_L}(s) = \frac{\dfrac{1}{\dfrac{1}{L\cdot s}+\dfrac{1}{R}}}{\dfrac{1}{\dfrac{1}{L\cdot s}+\dfrac{1}{R}}+\dfrac{1}{C\cdot s}}\cdot\mathbf{V_S}(s)$$

$$= \frac{1}{1+\dfrac{1}{C\cdot s}\cdot\left(\dfrac{1}{L\cdot s}+\dfrac{1}{R}\right)}\cdot\mathbf{V_S}(s) \quad = \frac{1}{1+\dfrac{1}{C\cdot s}\cdot\dfrac{1}{L\cdot s}+\dfrac{1}{C\cdot s}\cdot\dfrac{1}{R}}\cdot\mathbf{V_S}(s) \quad = \frac{s^2}{s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{L\cdot C}}\cdot\mathbf{V_S}(s)$$

$$\mathbf{H}(s) = \frac{\mathbf{V_L}(s)}{\mathbf{V_S}(s)} = \frac{s^2}{s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{L\cdot C}}$$

$$= \frac{s^2}{s^2+\dfrac{3.788\cdot 10^4}{sec}\cdot s+\dfrac{9.091\cdot 10^9}{sec^2}}$$

$R := 120\cdot\Omega \qquad C := 0.22\cdot\mu F \qquad L := 0.5\cdot mH$

$$\frac{1}{C\cdot R} = 3.788\cdot 10^4 \cdot\frac{1}{sec} \qquad \frac{1}{L\cdot C} = 9.091\cdot 10^9 \cdot\frac{1}{sec^2}$$

b) Find the characteristic equation for this circuit.

$$0 = s^2+\frac{1}{C\cdot R}\cdot s+\frac{1}{L\cdot C} = s^2+\frac{3.788\cdot 10^4}{sec}\cdot s+\frac{9.091\cdot 10^9}{sec^2}$$

Just the denominator set to zero. The solutions of the characteristic equation are the "poles" of the transfer function.

c) Find the differential equation for v_L.

Cross-multiply the transfer function

$$s^2\cdot\mathbf{V_S}(s) = \left(s^2+\frac{1}{C\cdot R}\cdot s+\frac{1}{L\cdot C}\right)\cdot\mathbf{V_L}(s)$$

$$s^2\cdot\mathbf{V_S}(s) = s^2\cdot\mathbf{V_L}(s)+\frac{1}{C\cdot R}\cdot s\cdot\mathbf{V_L}(s)+\frac{1}{L\cdot C}\cdot\mathbf{V_L}(s)$$

$$\frac{d^2}{dt^2}v_S(t) = \frac{d^2}{dt^2}v_L(t)+\frac{1}{C\cdot R}\cdot\frac{d}{dt}v_L(t)+\frac{1}{L\cdot C}\cdot v_L(t)$$

$$\frac{d^2}{dt^2}v_S(t) = \frac{d^2}{dt^2}v_L(t)+\frac{3.788\cdot 10^4}{sec}\cdot\frac{d}{dt}v_L(t)+\frac{9.091\cdot 10^9}{sec^2}\cdot v_L(t)$$

d) What are the solutions to the characteristic equation?

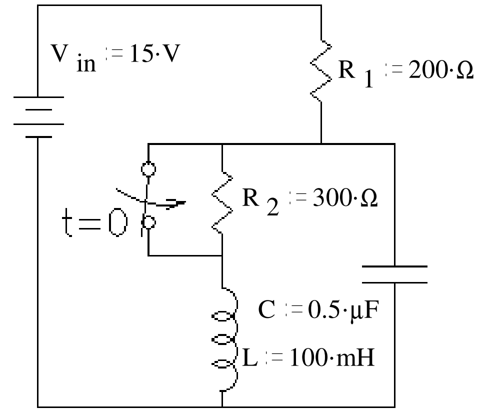
$$s_1 = \frac{-3.788\cdot 10^4}{2}+\frac{1}{2}\cdot\sqrt{\left(3.788\cdot 10^4\right)^2-4\cdot\left(9.091\cdot 10^9\right)} = -1.894\cdot 10^4 + 9.345\cdot 10^4 j$$

$$s_2 = \frac{-3.788\cdot 10^4}{2}-\frac{1}{2}\cdot\sqrt{\left(3.788\cdot 10^4\right)^2-4\cdot\left(9.091\cdot 10^9\right)} = -1.894\cdot 10^4 - 9.345\cdot 10^4 j$$

e) What type of response do you expect from this circuit?

The solutions to the characteristic equation are complex so the response will be **underdamped**.

2. Analysis of the circuit shown yields the characteristic equation below. The switch has been in the open position for a long time and is closed (as shown) at time t = 0. Find the initial and final conditions and write the full expression for $i_L(t)$, including all the constants that you find. Don't let the odd position of the switch throw you, just use it to find your initial conditions.
Clearly show important numbers (like initial and final conditions) to get partial credit. If you can't find some of these, guess so that you can move on.



$$s^2 + \left(\frac{1}{C\cdot R_1}\right)\cdot s + \left(\frac{1}{L\cdot C}\right) = 0$$

$$\left(\frac{1}{C\cdot R_1}\right) = 1\cdot 10^4 \cdot sec^{-1} \qquad \left(\frac{1}{L\cdot C}\right) = 4\cdot 10^9 \cdot sec^{-2}$$

$$s^2 + 10000\cdot\frac{1}{sec}\cdot s + 2\cdot 10^7\cdot\frac{1}{sec^2} = 0$$

$$s_1 := \left[\frac{-10000}{2} + \frac{1}{2}\cdot\sqrt{(10000)^2 - 4\cdot\left(2\cdot 10^7\right)}\right]\cdot sec^{-1} \qquad s_2 := \left[\frac{-10000}{2} - \frac{1}{2}\cdot\sqrt{(10000)^2 - 4\cdot\left(2\cdot 10^7\right)}\right]\cdot sec^{-1}$$

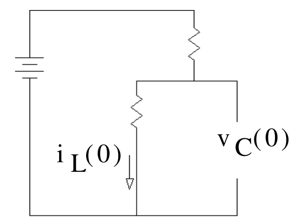
$$s_1 = -2764\cdot sec^{-1} \qquad s_2 = -7236\cdot sec^{-1}$$

s_1 and s_2 are both real and distinct, overdamped

Find the initial conditions:

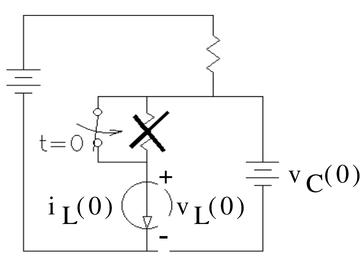
Before the switch closed, the inductor current was: $\frac{15\cdot V}{R_1 + R_2} = 30\cdot mA = i_L(0)$

Before the switch closed, the capacitor voltage was: $v_C(0) = \frac{R_2}{R_1 + R_2}\cdot(15\cdot V) = 9\cdot V$ so: $v_{C0} := 9\cdot V$



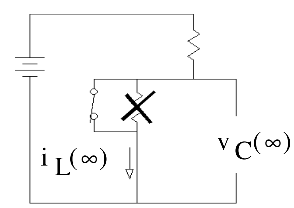
When the switch is opened, the new voltage across the inductor is: $v_{L0} := v_{C0}$

$$\frac{d}{dt}i_L(0) = \frac{1}{L}\cdot v_{L0} = \frac{1}{L}\cdot v_{C0} = 90\cdot\frac{A}{sec}$$



Find the final condition:

$$i_L(\infty) = \frac{15\cdot V}{R_1} = 75\cdot mA$$



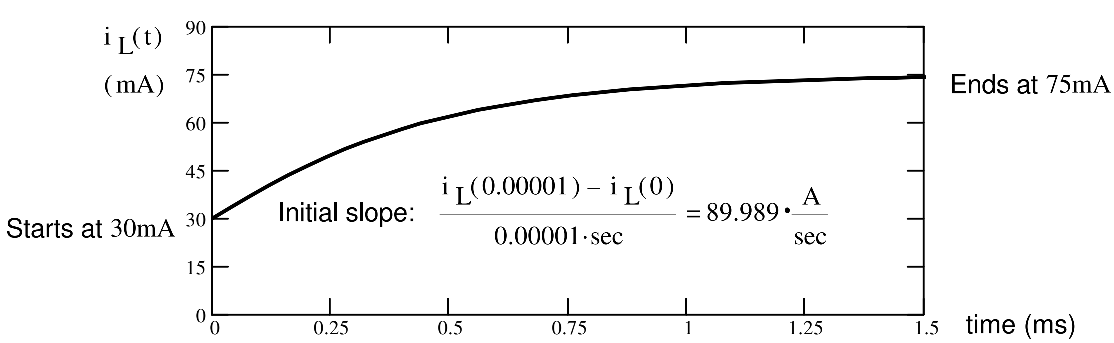
General solution for the overdamped condition: $i_L(t) = i_L(\infty) + B\cdot e^{s_1\cdot t} + D\cdot e^{s_2\cdot t}$

Initial conditions: $i_{L0} := \frac{15\cdot V}{R_1 + R_2} = i_L(\infty) + B + D$, so $B = i_{L0} - i_L(\infty) - D = 30\cdot mA - 75\cdot mA - D$
$= -45\cdot mA - D$

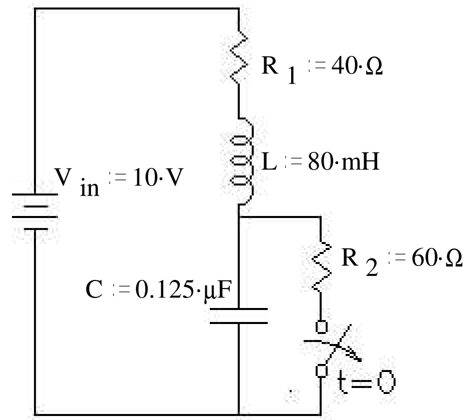
$$\frac{d}{dt}i_L(0) = 90\cdot\frac{A}{sec} = s_1\cdot B + s_2\cdot D = s_1\cdot(-45\cdot mA - D) + s_2\cdot D = s_1\cdot(-45\cdot mA) - s_1\cdot D + s_2\cdot D$$

solve for D & B: $D := \dfrac{90\cdot\frac{A}{sec} - s_1\cdot(-45\cdot mA)}{-s_1 + s_2}$ $\quad D = 7.69\cdot mA \quad B := -45\cdot mA - D \quad B = -52.7\cdot mA$

Plug numbers back in: $i_L(t) := 75\cdot mA - 52.7\cdot mA\cdot e^{-2764t} + 7.69\cdot mA\cdot e^{-7236t}$



3. Analysis of the circuit shown yields the characteristic equation and s values below. The switch has been in the closed position for a long time and is opened (as shown) at time $t = 0$. Find the initial and final conditions and write the full expression for $v_C(t)$, including all the constants.

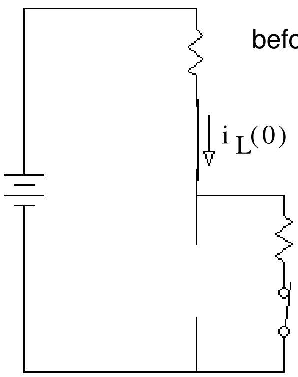


$$0 = s^2 + \frac{R_1}{L} \cdot s + \frac{1}{L \cdot C}$$

$$s_1 := \left(-250 + 10^4 \cdot j\right) \cdot \frac{1}{sec} \quad , \qquad s_2 := \left(-250 - 10^4 \cdot j\right) \cdot \frac{1}{sec}$$

Solution: $\alpha := -250 \cdot \frac{1}{sec}$ $\qquad \omega := 10000 \cdot \frac{rad}{sec}$

Initial conditions:

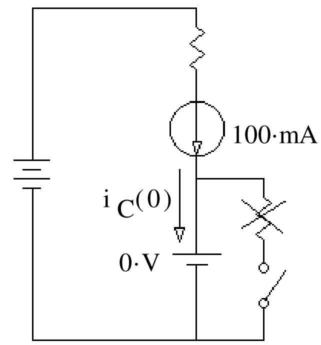


before switch opens

$$i_L(0) = \frac{V_{in}}{R_1 + R_2} = 100 \cdot mA$$

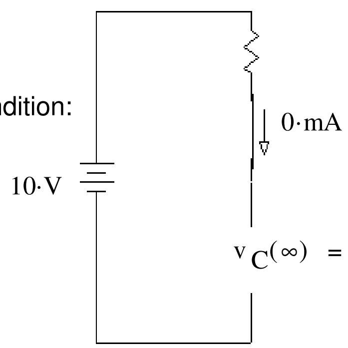
$$v_C(0) = V_{in} \cdot \frac{R_2}{R_1 + R_2} = 6 \cdot V$$

just after the switch opens



$$\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C} = \frac{100 \cdot mA}{C} = 8 \cdot 10^5 \cdot \frac{V}{sec}$$

Find final condition:



$$v_C(\infty) = V_{in} = 10 \cdot V$$

Find constants: $v_C(0) = v_C(\infty) + B$ $\qquad B = v_C(0) - v_C(\infty)$ $\qquad B := 6 \cdot V - 10 \cdot V$ $\qquad B = -4 \cdot V$

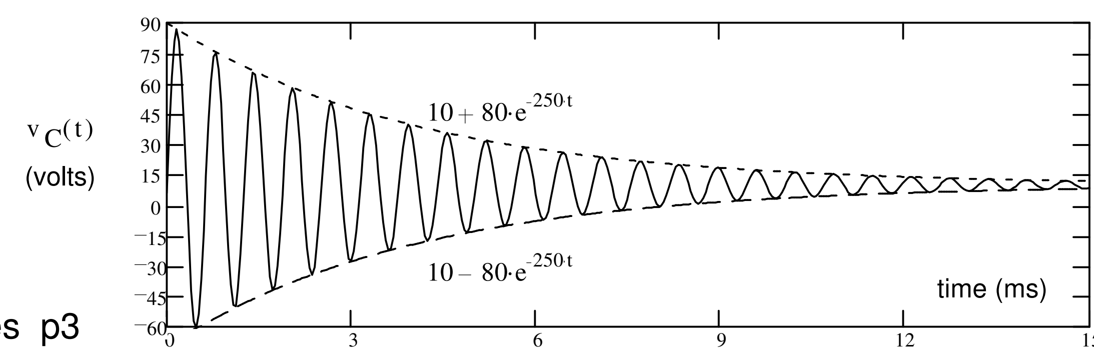
$$\frac{d}{dt} v_C(0) = \alpha \cdot B + D \cdot \omega \qquad D := \frac{8 \cdot 10^5 \cdot \frac{V}{sec} - \alpha \cdot B}{\omega} \qquad D = 79.9 \cdot V$$

Write the full expression for $v_C(t)$, including all the constants that you find.

$$v_C(t) = e^{\alpha t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t)) + v_C(\infty)$$

$$v_C(t) := e^{-250 t} \cdot \left(-4 \cdot V \cdot \cos\left(10^4 \cdot t\right) + 79.9 \cdot V \cdot \sin\left(10^4 \cdot t\right)\right) + 10 \cdot V$$

$$\sqrt{D^2 + B^2} = 80 \cdot V$$



4. Analysis of a circuit (not pictured) yields the characteristic equation below.

$$0 = s^2 + 400\cdot s + 400000 \qquad R := 80\cdot\Omega \qquad L := 20\cdot mH \qquad C := 2\cdot\mu F$$

Further analysis yields the followiing initial and final conditions:

$i_L(0) = 120\cdot mA$ $\quad v_L(0) = -3\cdot V$ $\quad v_C(0) = 7\cdot V$ $\quad i_C(0) = -80\cdot mA$

$i_L(\infty) = 800\cdot mA$ $\quad v_L(\infty) = 0\cdot V$ $\quad v_C(\infty) = 12\cdot V$ $\quad i_C(\infty) = 0\cdot mA$

Write the full expression for $i_L(t)$, including all the constants that you find. $\quad i_L(t) = ?$

Solution: $\dfrac{400}{2} = 200$ $\qquad \dfrac{\sqrt{400^2 - 4\cdot 400000}}{2} = 600j$

$s_1 := (-200 + 600\cdot j)\cdot\dfrac{1}{sec}$ and $s_2 := (-200 - 600\cdot j)\cdot\dfrac{1}{sec}$

$\alpha := \mathrm{Re}(s_1)$ $\quad \alpha = -200\cdot sec^{-1}$ $\qquad \omega := \mathrm{Im}(s_1)$ $\quad \omega = 600\cdot sec^{-1}$

Initial slope: $\dfrac{d}{dt}i_L(0) = \dfrac{v_L(0)}{L} = \dfrac{-3\cdot V}{L} = -150\cdot\dfrac{A}{sec}$

General solution for the underdamped condition: $i_L(t) = i_L(\infty) + e^{\alpha\cdot t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t))$

Find constants: $i_L(0) = i_L(\infty) + B$ $\qquad B = i_L(0) - i_L(\infty)$ $\qquad B := 120\cdot mA - 800\cdot mA$

$B = -680\cdot mA$

$\dfrac{d}{dt}i_L(0) = \alpha\cdot B + D\cdot\omega$ $\qquad D := \dfrac{-150\cdot\frac{A}{sec} - \alpha\cdot B}{\omega}$ $\qquad D = -476.667\cdot mA$

Write the full expression for $i_L(t)$, including all the constants that you find.

$$i_L(t) := 800\cdot mA + e^{-200t}\cdot(-680\cdot mA\cdot\cos(600\cdot t) - 477\cdot mA\cdot\sin(600\cdot t))$$