Ex:

\[
\begin{align*}
R_1 &= 10 \text{ k}\Omega \\
R_2 &= 30 \text{ k}\Omega \\
L &= 40 \text{ nH} \\
C &= 10 \text{ nF} \\
R_3 &= 30 \text{ k}\Omega \\
\end{align*}
\]

a) What type of filter is the above circuit: a band-pass or a band-reject?

**Hint:** Use a Thevenin equivalent to combine all the R's into one.

For the filter shown above, calculate the following quantities:

b) \(\omega_o\)  

c) \(\omega_{C1}\) and \(\omega_{C2}\)  

d) \(\beta\) and \(Q\)

**SOL'N:** a) We convert to the frequency domain and use a voltage divider in terms of phasors and impedances. Then we move \(R_3\) to the left of \(L\) and \(C\) and take the Thevenin equivalent of \(V_i, R_1, R_2,\) and \(R_3\).

\[
R_{Th} = R_1 || R_2 || R_3
\]

\[
\frac{V_i}{V_o} = \frac{R_2 || R_3}{R_1 + R_2 || R_3}
\]

\[
\frac{1}{j\omega C}
\]

This is a band-reject filter.

At \(\omega = 0\), the capacitor is an open circuit and the input signal is connected to the output by \(R_{Th}\), (albeit the magnitude is attenuated by the voltage divider multiplying \(V_i\)).
At $\omega \to \infty$, the inductor is an open circuit and the input signal is connected to the output by $R_{Th}$, (albeit the magnitude is attenuated by the voltage divider multiplying $V_i$).

At $\omega_0 = 1/\sqrt{LC}$, the inductor's and capacitor's impedances will sum to zero, which is equivalent to a wire. The output voltage is shorted to zero, and this frequency is totally rejected, (i.e., doesn't reach the output).

We have now answered part (a), but we derive the transfer function of the filter in preparation for parts (b) thru (d).

This circuit is a voltage divider:

$$ V_o = V_i \frac{R_2 \parallel R_3}{R_1 + R_2 \parallel R_3} \cdot \frac{j\omega L + 1/j\omega C}{R_{Th} + j\omega L + 1/j\omega C} $$

or

$$ H(j\omega) = \frac{V_o}{V_i} = \frac{R_2 \parallel R_3}{R_1 + R_2 \parallel R_3} \cdot \frac{1}{1 + \frac{R_{Th}}{j\omega L + 1/j\omega C}} $$

or

$$ H(j\omega) = \frac{V_o}{V_i} = \frac{R_2 \parallel R_3}{R_1 + R_2 \parallel R_3} \cdot \frac{1}{1 + \frac{R_{Th}}{j(\omega L - 1/\omega C)}} $$

or

$$ H(j\omega) = \frac{V_o}{V_i} = \frac{R_2 \parallel R_3}{R_1 + R_2 \parallel R_3} \cdot \frac{1}{1 - j \frac{R_{Th}}{(\omega L - 1/\omega C)}} $$

b) The center frequency is, as always, given by the following formula:

$$ \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1 \text{ r/s}}{\sqrt{40n \cdot 10n}} = \frac{1 \text{ r/s}}{20n} = 50 \text{ Mr/s} $$

c) The cutoff frequency is, by definition, the frequency where the magnitude response is equal to the maximum of the magnitude response divided by the square root of two.
For $\omega = 0$ or $\infty$, we have a maximum value for the magnitude of $H$:

$$\max|H(j\omega)| = \frac{R_2 \ || \ R_3}{R_1 + R_2 \ || \ R_3} \cdot \frac{1}{1 - j \frac{R_{Th}}{\infty}} = \frac{R_2 \ || \ R_3}{R_1 + R_2 \ || \ R_3}$$

Thus, we solve the following for $\omega$:

$$|H(j\omega)| = \frac{R_2 \ || \ R_3}{R_1 + R_2 \ || \ R_3} \cdot \frac{1}{1 - j \frac{R_{Th}}{(\omega L - 1/\omega C)}} = \frac{1}{\sqrt{2}} \frac{R_2 \ || \ R_3}{R_1 + R_2 \ || \ R_3}$$

or

$$\frac{1}{1 - j \frac{R_{Th}}{(\omega L - 1/\omega C)}} = \frac{1}{\sqrt{2}}$$

The solution of this equation occurs when the imaginary part of the denominator is equal to plus or minus one, since $|1 \pm j| = \sqrt{2}$.

$$\frac{R_{Th}}{(\omega L - 1/\omega C)} = \pm 1$$

This gives two quadratic equations to solve

$$R_{Th} = \pm (\omega L - 1/\omega C)$$

or

$$\omega R_{Th} = \pm (\omega^2 L - 1/C)$$

or

$$\pm \omega R_{Th} = \omega^2 L - 1/C$$

or

$$\omega^2 \pm \omega \frac{R_{Th}}{L} - 1/LC = 0$$

Using the two positive roots, we have the cutoff frequencies:

$$\omega_{C1,2} = \pm \frac{R_{Th}}{2L} + \sqrt{\left(\frac{R_{Th}}{2L}\right)^2 + \frac{1}{LC}}$$
Using circuit values, we have the following:

\[ R_{Th} = 10k \parallel 30k \parallel 30k = 10k \parallel 15k = 6 \, k\Omega \]

and

\[ \frac{R_{Th}}{2L} = \frac{6k}{2(40n)} = 75 \, Gr/s \]

Inside the square root, we have the \( R_{Th} \) term is much larger than the \( \omega_o \) term. This allows us to use an approximation:

\[ \sqrt{1+x} \approx 1 + \frac{x}{2} \quad \text{when } 0 \leq x \ll 1 \]

Thus, we have the following:

\[
\omega_{C1,2} = \pm \frac{R_{Th}}{2L} + \frac{R_{Th}}{2L} \sqrt{1 + \frac{1/LC}{\left(\frac{R_{Th}}{2L}\right)^2}} = \frac{R_{Th}}{2L} \left[ \pm 1 + \frac{1}{2} \frac{1/LC}{\left(\frac{R_{Th}}{2L}\right)^2} \right]
\]

or

\[
\omega_{C1,2} = 75 \, Gr/s \left[ \pm 1 + \frac{1}{2} \left(\frac{50 \, M}{75 \, G}\right)^2 \right] = 75 \, Gr/s \left[ \pm 1 + \frac{1}{2} \left(\frac{4}{9}\right) \right]
\]

or

\[
\omega_{C1} \approx \frac{2}{9} \, kr/s = 16.7 \, kr/s
\]

\[
\omega_{C2} = 150 \, Gr/s
\]

d) The bandwidth, \( \beta \), is the difference of the cutoff frequencies:

\[ \beta = \omega_{C2} - \omega_{C1} = \frac{R_{Th}}{L} = 150 \, Gr/s \]

The quality factor, \( Q \), is the center frequency over the bandwidth:

\[ Q = \frac{\omega_o}{\beta} = \frac{50 \, Mr/s}{150 \, Gr/s} = \frac{1}{3} \, m \quad \text{(very low!)} \]