Ex: Find the inverse Laplace transform for the following expression:

\[
F(s) = \frac{6s^2 + 36s + 198}{(s + 3)(s^2 + 6s + 45)}
\]

SOL’N: One approach to this problem is to use a partial fraction expansion with a coefficient for every root. Because the constant term of the quadratic in the denominator is larger than the square of half the middle coefficient, the quadratic has complex roots.

\[
s^2 + 6s + 45 = (s + 3 + j6)(s + 3 - j6) = (s + a + j\omega)(s + a - j\omega)
\]

The coefficients for the complex roots will be complex conjugates.

\[
F(s) = \frac{6s^2 + 36s + 198}{(s + 3)(s^2 + 6s + 45)} = \frac{A}{s + 3} + \frac{B}{s + 3 + j6} + \frac{B^*}{s + 3 - j6}
\]

We multiply by a root term and then evaluate at the root value. For the first root, we have the following calculation:

\[
A = (s + 3)F(s)\bigg|_{s=-3} = \bigg. \frac{6s^2 + 36s + 198}{s^2 + 6s + 45}\bigg|_{s=-3}
\]

or

\[
A = \frac{6(-3)^2 + 36(-3) + 198}{(-3)^2 + 6(-3) + 45} = \frac{54 - 108 + 198}{9 - 18 + 45} = \frac{144}{36} = 4
\]

The inverse transform for this term is an exponential decay:

\[
\mathcal{L}^{-1}\left[\frac{4}{s + 3}\right] = 4e^{-3t}u(t)
\]

For the second root, we have the following calculation:

\[
B = (s + 3 + j6)F(s)\bigg|_{s=-3-j6} = \bigg. \frac{6s^2 + 36s + 198}{(s + 3)(s + 3 - j6)}\bigg|_{s=-3-j6}
\]

or

\[
B = \frac{6(-3 - j6)^2 + 36(-3 - j6) + 198}{(-3 - j6 + 3)(-3 - j6 + 3 - j6)} = \frac{6\left[(-3 - j6)^2 + 6(-3 - j6) + 33\right]}{-j6(-j12)}
\]
We can cancel a factor of 6 on top and bottom:

\[
B = \frac{(-3 - j6)^2 + 6(-3 - j6) + 33}{-12} = \frac{9 + j36 - 36 + -18 - j36 + 33}{-12}
\]

or

\[
B = \frac{-12}{-12} = 1
\]

Since this coefficient is real, its complex conjugate has the same value:

\[B^* = 1\]

Using the complex root identity for decaying \(\cos()\) and \(\sin()\), the inverse transform for the complex root terms are as follows:

\[
L^{-1}\left[\frac{1}{s + 3 + j6} + \frac{1}{s + 3 - j6}\right] = 2e^{-3t}\cos(6t)u(t)
\]

Combining the above results, we have our final answer:

\[
L^{-1}[F(s)] = \left[4e^{-3t} + 2e^{-3t}\cos(6t)\right]u(t)
\]

Another approach to this problem is to replace \(s + 3\) with \(s\) or, equivalently, replace \(s\) with \(s - 3\) and multiply in the time domain by \(e^{-3t}\) in the final step. In other words, this approach exploits the "multiply by \(e^{-at}\)" identity:

\[
L\left[e^{-at}v(t)\right] = V(s)|_{s+a} \text{ replaces } s
\]

So we replace \(s + 3\) with \(s\):

\[
F_2(s) = \frac{6(s - 3)^2 + 36(s - 3) + 198}{s(s^2 + 36)}
\]

We can factor out a 6 from the top to make the numbers smaller in our calculations:

\[
F_2(s) = \frac{6\left[(s - 3)^2 + 6(s - 3) + 33\right]}{s(s^2 + 36)}
\]
Now we work on this expression and match the complex root terms to a cosine and sine:

\[ F_2(s) = \frac{6[(s - 3)^2 + 6(s - 3) + 33]}{s(s^2 + 36)} = \frac{A}{s} + \frac{Bs}{s^2 + 6^2} + \frac{C\omega}{s^2 + 6^2} \]

Recall the transform pairs for cosine and sine are as follows:

\[ \mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + 6^2} \quad \text{and} \quad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + 6^2} \]

We can find \( A \) by the usual method of multiply by the root term and evaluating at the root value:

\[ A = sF_2(s) \bigg|_{s=0} = \frac{6[(s - 3)^2 + 6(s - 3) + 33]}{(s^2 + 36)} \bigg|_{s=0} = \frac{6[(-3)^2 + 6(-3) + 33]}{36} \]

or

\[ A = \frac{6[9 - 18 + 33]}{36} = \frac{6 \cdot 24}{36} = 4 \]

To find \( B \) and \( C \), we can use a common denominator.

\[ F_2(s) = \frac{6[(s - 3)^2 + 6(s - 3) + 33]}{s(s^2 + 36)} = \frac{A(s^2 + 6^2) + Bs^2 + C\omega}{s(s^2 + 6^2)} \]

Starting with the highest power of \( s \) in the numerator, we match coefficients of each power of \( s \):

\[ 6s^2 = As^2 + Bs^2 = 4s^2 + Bs^2 \]

The value of \( B \) is easily computed:

\[ B = 2 \]

We find the value of \( C \) by matching the coefficient of \( s \) in the numerator:

\[ 6[-6s + 6s] = C\omega \]

We conclude that \( C \) is zero:

\[ C = 0 \]
Now we have the partial fraction expansion for $F_2(s)$:

$$F_2(s) = \frac{4}{s} + \frac{2s}{s^2 + 6^2}$$

Taking the inverse transform and multiplying by $e^{-3t}$, we have our final answer, which is the same as before:

$$\mathcal{L}^{-1}[F(s)] = [4e^{-3t} + 2e^{-3t} \cos(6t)]u(t)$$

**Ack:** The author gratefully acknowledges Leroy Pimental for suggesting the second approach.