Ex:

After being closed for a long time, the switch opens at $t = 0$.

a) State whether $i(t)$ is underdamped, overdamped, or critically damped.

b) Write a numerical time-domain expression for $i(t)$, $t > 0$, the current through $C$. This expression must not contain any complex numbers.

**Sol’N:** a) We find the characteristic roots of the RLC circuit after time $t = 0$:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where

$$\alpha = \frac{R}{2L} \text{ (series RLC) and } \omega_0^2 = \frac{1}{LC}$$

Using values from the circuit, we have the following result:

$$\alpha = \frac{0.1 \ \Omega}{2 \cdot 3 \ \mu H} = \frac{100 \ \text{kr/s}}{6} = 16.67 \ \text{kr/s}$$

and

$$\omega_0^2 = \frac{1}{3 \ \mu H \cdot 1200 \ \mu F} = \left(\frac{1 \ \text{Mr/s}}{60}\right)^2 = \left(\frac{100 \ \text{kr/s}}{6}\right)^2$$

Since $\alpha = \omega_0$, the square root is zero and the circuit is critically damped:

$$s_1 = s_2.$$
b) First, we note that the form of our solution is as follows:

\[ i(t) = A_1 e^{-\alpha t} + A_2 te^{-\alpha t} + A_3 \]

where \( s = -\alpha = -\frac{100 \text{ kr/s}}{6} \)

Second, we find \( A_3 \), which is equal to the final value of \( i(t) \). As \( t \to \infty \), the switch is open, eliminating the 0.2 \( \Omega \) resistor and 12 V source. Also, the \( L \) acts like a wire and the \( C \) acts like an open circuit. This leaves us with a wire, a 0.1 \( \Omega \) resistor, in a loop around the outside with an open at the bottom. All currents and voltages in this circuit will be zero.

\[ A_3 = i(t \to \infty) = 0 \text{ A} \]

Third, we find the initial conditions in the circuit and match them to initial values for our symbolic solution:

\[ i(t = 0^+) = A_1 \text{ and } \left. \frac{d i(t)}{dt} \right|_{t=0^+} = -\alpha A_1 + A_2 \]

For the circuit, we consider the energy variables \( i_L \) and \( v_C \) at \( t = 0^- \), when the \( L \) acts like a wire and the \( C \) acts like an open circuit. Also, the switch is closed.

In this case, the inductor is a short and the 0.1 \( \Omega \) resistor is dangling at the end of a wire. Since no current flows through the 0.1 \( \Omega \) resistor, the voltage across the \( C \) equals the voltage across the \( L \), which is zero:

\[ v_C(0^-) = 0 \text{ V} \]

The upper part of the circuit forms a loop with current, (measured with arrow pointing to the right), determined by the voltage source and resistor.

\[ i_L(0^-) = -\frac{12 \text{ V}}{0.2 \Omega} = -60 \text{ A} \]

Moving to time \( t = 0^+ \), these energy values don't change. We treat the \( C \) as a voltage source of zero volts and the \( L \) as a current source of –60 Amps. Also, the switch is now open.

Since the \( C \) is now in series with the \( L \), we have the initial value of \( i \):

\[ i(0^+) = i_L(0^+) = i_L(0^-) = -60 \text{ A} \]
This gives us the value of $A_1$:

$$A_1 = i(0^+) = -60 \text{ A}$$

Finally, we want to find the value of the derivative of $i$ at time $t = 0^+$. To do so, we equate $i$ to an expression in terms of energy variables, $i_L$ and/or $v_C$. Since $i = i_L$, our work is done:

$$i(t) = i_L(t)$$

We differentiate this equation to obtain an expression for $i$ having only energy (or state) variables on the right side. Using the basic component equations for an $L$ and $C$, we can express the right side in terms of non-derivatives:

$$\frac{di(t)}{dt} \bigg|_{t=0^+} = \frac{di_L(t)}{dt} \bigg|_{t=0^+} = \frac{v_L(0^+)}{L}$$

To find the initial voltage on the $L$, we write this voltage, (+ on left, – on right), as a function of the initial values of the energy variables, $i_L$ and $v_C$.

$$v_L(0^+) = -v_C(0^+) - i_L(0^+)R \text{ where } R = 0.1 \Omega$$

or

$$v_L(0^+) = 0 - 60 \text{ A} \cdot 0.1 \Omega = 6 \text{ V}$$

Thus, we have the following result:

$$\frac{di(t)}{dt} \bigg|_{t=0^+} = \frac{v_L(0^+)}{L} = \frac{6 \text{ V}}{3 \mu\text{H}} = 2 \text{ MA/s} = -\alpha A_1 + A_2$$

Using values from earlier, we have the following result:

$$A_2 = 2 \text{ MA/s} + \frac{100 \text{ kr/s}}{6}(-60 \text{ A}) = 1 \text{ MA/s}$$

This yields the expression for $i(t)$:

$$i(t > 0) = -60e^{-16.67\text{kr/s}t} + 1 \text{ M} te^{-16.67\text{kr/s}t} \text{ A}$$