2.

One period, $T$, of a function $v(t)$ is shown above. The formula for $v(t)$ is

$$v(t) = \begin{cases} 
3 + \frac{3t}{T/8} & 0 < t < T/8 \\
0 & T/8 < t < T/4 \\
3 & T/4 < t < 3T/4 \\
6 & 3T/4 < t < 7T/8 \\
\frac{3(t - 7T/8)}{T/8} & 7T/8 < t < T
\end{cases}$$

Find the numerical value of the following coefficients of the Fourier series for $v(t)$:
(Hint: remove the DC offset after finding its value.)

a) $a_0$  
b) $a_1$

3. a) Find the value of $a_2$ and $b_1$ for the above Fourier series.
**SOL’N:** a) By inspection we can smooth out the function to find its average value. We observe that the triangular spike on the left may be cut off at 3V, rotated 180°, and placed on top of the triangular divot on the right side to yield flat sections of height 3V on the far left and far right. Similarly, the block poking up just after \( t = 3T/4 \) may be cut off at 3V and placed in the square hole between \( T/8 \) and \( T/4 \) to yield flat sections of height 3V where the block and hole were. Thus, the height of the resulting surface has a constant value of 3V, and this must be the average height of the waveform:

\[ a_\nu = 3 \text{ V} \]

b. 3) Having found the average value (or DC offset) of the waveform, we may subtract it out for the calculation of the remaining terms of the Fourier series. This yields an odd function:

\[ v(t) = \begin{cases} 
6V & \text{for } 0 \leq t < T/8 \\
3V & \text{for } T/8 \leq t < T/4 \\
0 & \text{for } T/4 \leq t < 3T/4 \\
-3V & \text{for } 3T/4 \leq t < 3T/8 \\
-6V & \text{for } 3T/8 \leq t < T 
\end{cases} \]

For an odd function, all coefficients of cosine terms are zero:

\[ a_1 = 0 \text{ V} \quad \text{and} \quad a_2 = 0 \text{ V} \]
b) For the $b_1$ coefficient, we begin by sketching the functions involved in the integral formula for the $b_1$ calculation:

$$b_1 = \frac{2}{T} \int_0^T [v(t) - 3V] \sin(\omega_0 t) \, dt$$

We observe that the area under the product curve in the first half of the period has the same shape and sign as the area under the product curve in the second half of the period. Thus, we calculate the area for the first half and double it. We use piecewise definitions of the voltage waveform to do this:

$$b_1 = 2 \cdot \frac{2}{T} \left[ \int_0^{T/8} \frac{3V}{T/8} \sin(\omega_0 t) \, dt + \int_{T/8}^{T/4} (-3V) \sin(\omega_0 t) \, dt \right]$$

or

$$b_1 = 2 \cdot \frac{2}{T} \left[ \frac{24V}{T} \int_0^{T/8} t \sin(\omega_0 t) \, dt - 3V \int_{T/8}^{T/4} \sin(\omega_0 t) \, dt \right]$$

Because $T$ disappears in the final answer, we may leave it as a symbolic variable or use any value we like for $T$. An interesting choice is to let $T = 2\pi$ so $\omega_0 = 1$:

$$b_1 = 2 \cdot \frac{2}{2\pi} \left[ \frac{24V}{2\pi} \int_0^{2\pi/8} t \sin(t) \, dt - 3V \int_{2\pi/8}^{2\pi/4} \sin(t) \, dt \right]$$

Using integral tables or a calculator, we have the following result:

$$b_1 = 2 \cdot \frac{2\pi}{2\pi} \left[ 12V \left( \sin t - t \cos t \right) \bigg|_{0}^{2\pi/8} - 3V \left( -\cos t \right) \bigg|_{2\pi/8}^{2\pi/4} \right]$$

or

$$b_1 = 2 \cdot \frac{2\pi}{\pi} \left[ 12V \left( \sin \frac{\pi}{4} - \frac{\pi}{4} \cos \frac{\pi}{4} \right) - (\sin 0 - 0 \cos 0) - 3V \left( -\cos \frac{\pi}{2} + \cos \frac{\pi}{4} \right) \right]$$

or
\[ b_1 = \frac{2}{\pi} \left[ \frac{12V}{\pi} \left( \sin \frac{\pi}{4} - \frac{\pi}{4} \cos \frac{\pi}{4} \right) - 3V \left( \cos \frac{\pi}{4} \right) \right] \]

or

\[ b_1 = \frac{2}{\pi} \left[ \frac{12V}{\pi} \left( \frac{\sqrt{2}}{2} \left[ 1 - \frac{\pi}{4} \right] \right) - 3V \left( \frac{\sqrt{2}}{2} \right) \right] \]

or

\[ b_1 = \frac{1}{\pi} \left[ \frac{12V}{\pi} \left( \sqrt{2} \left[ 1 - \frac{\pi}{4} \right] \right) - 3V (\sqrt{2}) \right] \]

or

\[ b_1 = \frac{12\sqrt{2}}{\pi^2} - \frac{6\sqrt{2}}{\pi} V = \frac{6\sqrt{2}}{\pi} \left( \frac{2}{\pi} - 1 \right) V \approx -0.98 \text{ V} \]