Ex: Find the inverse Laplace transform for the following expression:

\[ F(s) = \frac{2s - 26}{s^2 + 10s + 169} \]

Sol'n: Our first step in finding the inverse transform is to express the denominator in terms of roots. For a quadratic polynomial, if the square of half the coefficient of \( s \) is less than the constant coefficient, the roots are complex. In that case, we can write the denominator in terms of the real and imaginary parts of the roots:

\[ s^2 + 10s + 169 = (s + a)^2 + \omega^2 = s^2 + 2as + a^2 + \omega^2 \]

In this expression, \( a \) is the real part of the root and \( \omega \) is the imaginary part of the root. From this expression, we see that \( a \) equals half the middle coefficient:

\[ a = \frac{10}{2} = 5 \]

To find \( \omega \), we use the value of \( a \) and the constant term in the denominator:

\[ a^2 + \omega^2 = 169 \]
\[ 5^2 + \omega^2 = 169 \]
\[ \omega = \pm \sqrt{169 - 5^2} = \pm \sqrt{144} = \pm 12 \]

Our roots are complex conjugates:

\[ s_{1,2} = -a \pm j\omega = 5 \pm j12 \]

We can express the denominator in several ways:

\[ s^2 + 10s + 169 = (s - (-a + j\omega))(s - (-a - j\omega)) = (s + a - j\omega)(s + a + j\omega) \]
\[ s^2 + 10s + 169 = (s + 5 - j12)(s + 5 + j12) \]

or

\[ s^2 + 10s + 169 = (s + a)^2 + \omega^2 \]
\[ s^2 + 10s + 169 = (s + 5)^2 + 12^2 \]

If we choose the first form for the denominator, we express \( F(s) \) as partial fractions:
\[ F(s) = \frac{A_1}{s + 5 - j12} + \frac{A_1^*}{s + 5 + j12} \]

**NOTE:** Because the roots are conjugates and all the coefficients in \( F(s) \) are real, the coefficients of the partial fractions are always complex conjugates of each other.

We find \( A_1 \) by multiplying \( F(s) \) by the root term and evaluating at the value of the root.

\[
A_1 = (s + 5 - j12)F(s) \bigg|_{s = -(5 - j12)} = \frac{2s - 26}{s + 5 + j12} \bigg|_{s = -(5 - j12)}
\]

or

\[
A_1 = \frac{2[-(5 - j12)] - 26}{-(5 - j12) + 5 + j12} = \frac{-36 + j24}{j24} = \frac{(-j)(-36 + j24)}{24} = 1 - j \cdot \frac{3}{2}
\]

**NOTE:** The value in the denominator will always be two times the imaginary part of the root we are evaluating, as the real parts will cancel out.

If we use a common denominator, we can identify terms for a decaying cosine and sine.

\[ F(s) = \frac{1 + j \cdot \frac{3}{2}}{s + 5 - j12} + \frac{1 - j \cdot \frac{3}{2}}{s + 5 + j12} \]

or

\[ F(s) = \frac{\left(1 + j \cdot \frac{3}{2}\right)(s + 5 + j12) + \left(1 - j \cdot \frac{3}{2}\right)(s + 5 - j12)}{(s + 5 - j12)(s + 5 + j12)} \]

or

\[ F(s) = \frac{2s + 2 \cdot 5 - 2 \cdot \frac{3}{2} \cdot 12}{s^2 + 10s + 169} = \frac{2s - 26}{s^2 + 10s + 169} \]
Symbolically, if we write $A_1$ as a complex number, we can express our results in generic form.

$$A_1 = c + jd$$

$$F(s) = \frac{c + jd}{s + a - j\omega} + \frac{c - jd}{s + a + j\omega} = \frac{2cs + 2ca - 2d\omega}{(s + a)^2 + \omega^2}$$

Although it appears we have simply come full circle back to our original expression for $F(s)$, we can ultimately express our results in terms of $A_1$.

We observe that the denominator is the denominator of a decaying cosine or sine. We now represent $F(s)$ as a sum of transforms for a decaying cosine and sine:

$$F(s) = \frac{K_1(s + a)}{(s + a)^2 + \omega^2} + \frac{K_2\omega}{(s + a)^2 + \omega^2}$$

Equating the numerators with the numerator of the previous expression yields expression for $K_1$ and $K_2$:

$$2cs + 2ca - 2d\omega = K_1(s + a) + K_2\omega$$

Matching the coefficient for the highest power of $s$ first yields our result in terms of the real and complex parts of $A_1$:

$$K_1 = 2c \quad \text{and} \quad K_2 = -2d$$

**NOTE:** We can also bypass the steps of finding $A_1$ and equate the numerator of $F(s)$ directly with $K_1(s + a) + K_2\omega$. We see that $K_1$ is the coefficient of $s$. Once we find $K_1$, we solve for $K_2$:

$$K_2 = \frac{\text{constant term} - K_1a}{\omega}$$

Here, we will have $K_1 = 2$ and $K_2 = -3$. This approach is possible, however, only if we have an expression that has only two roots. If there are more roots, we must find $A_1$.

Now we take the inverse transform:
\[ f(t) = \mathcal{L}^{-1}\left\{ \frac{K_1(s + a)}{(s + a)^2 + \omega^2} \right\} + \mathcal{L}^{-1}\left\{ \frac{K_2\omega}{(s + a)^2 + \omega^2} \right\} \]

or

\[ f(t) = K_1e^{-at}\cos(\omega t) + K_2e^{-at}\sin(\omega t) \]

or

\[ f(t) = 2ce^{-at}\cos(\omega t) - 2de^{-at}\sin(\omega t) \]

or

\[ f(t) = 2\text{Re}[A_1]e^{-at}\cos(\omega t) - 2\text{Im}[A_1]e^{-at}\sin(\omega t) \]

**Note:** $A_1$ is the coefficient of the root term in the denominator that has a minus sign in it. If we find the coefficient of the root term in the denominator that has a plus sign in it, then we must use the conjugate of $A_1$ in the above expression. This changes the sign of the decaying sine term. (The cosine term is unaffected.)

Here, these formulas give our final result:

\[ f(t) = 2e^{-5t}\cos(12t) - 3e^{-5t}\sin(12t) \]