Ex: Find the inverse Laplace transform for the following expression:

\[ F(s) = \frac{9s^2 + 35s + 49}{(s + 2)(s^2 + 2s + 5)} \]

SOL’N: We first find the root terms of the quadratic term in the denominator. Since the square of one-half the middle term equals 1², which is less than the constant term, 5, the roots are complex.

We find the roots, \( s_{1,2} = -a \pm j\omega \), as follows:

\[ s^2 + 2s + 5 = (s + a)^2 + \omega^2 = s^2 + 2as + a^2 + \omega^2 \]

The value of \( a \) is half the middle coefficient:

\[ a = \frac{2}{2} = 1 \]

Using this value of \( a \) and the constant term of the quadratic, we find the value of \( \omega \):

\[ \omega = \sqrt{5 - a^2} = \sqrt{5 - 1} = 2 \]

Our partial fraction expansion is as follows:

\[ F(s) = \frac{A_1}{s + 2} + \frac{A_2}{s + 1 - j2} + \frac{A_2^*}{s + 1 + j2} \]

We multiply by root terms and evaluate at roots to find the partial fraction coefficients:

\[ A_1 = (s + 2)F(s) \bigg|_{s=-2} = \frac{9s^2 + 35s + 49}{s^2 + 2s + 5} \bigg|_{s=-2} = \frac{9 \cdot 4 - 35 \cdot 2 + 49}{4 - 2 \cdot 2 + 5} \]

or

\[ A_1 = \frac{15}{5} = 3 \]

The \( A_2 \) coefficient is complex:

\[ A_2 = (s + 1 - j2)F(s) \bigg|_{s=(1-j2)} = \frac{9s^2 + 35s + 49}{(s + 2)(s + 1 + j2)} \bigg|_{s=(1-j2)} \]

or
\[
A_2 = \frac{9((1-j2)^2 - 35(1-j2) + 49)}{(-(1-j2) + 2)[-(1-j2) + 1+j2]} = \frac{9(-3-j4) - 35(1-j2) + 49}{(1+j2)j4}
\]

or
\[
A_2 = \frac{-13 + j34}{-8 + j4} = \frac{-13 + j34}{-8 + j4}, \quad \frac{-8 - j4}{-8 - j4} = \frac{104 + 136 + j(52 - 272)}{80}
\]

or
\[
A_2 = \frac{240 - j220}{80} = 3 - \frac{j11}{4}
\]

We now have the following expansion:
\[
F(s) = \frac{3}{(s+2)} + \frac{3 - \frac{j11}{2}}{s+1-j2} + \frac{3 + \frac{j11}{2}}{s+1+j2}
\]

For the complex poles we use the following identity:
\[
\mathcal{L}^{-1}\left\{ \frac{c + jd}{s+a-j\omega} + \frac{c-jd}{s+a+j\omega} \right\} = 2ce^{-at}\cos(\omega t) - 2de^{-at}\sin(\omega t)
\]

Our final result is a decay plus a decaying cosine and decaying sine:
\[
f(t) = 3e^{-2t} + 6e^{-t}\cos(2t) + \frac{11}{2}e^{-2t}\sin(2t)
\]

An alternative approach is to find \(A_1\) and then write the original expansion in terms of the decaying cosine and decaying sine transforms:
\[
F(s) = \frac{A_1}{(s+2)} + \frac{K_2(s+a)}{s^2 + 2s + 5} + \frac{K_3 \cdot \omega}{s^2 + 2s + 5}
\]

or
\[
F(s) = \frac{3}{(s+2)} + \frac{K_2(s+1)}{s^2 + 2s + 5} + \frac{K_3 \cdot 2}{s^2 + 2s + 5}
\]

We then put everything over a common denominator and match the numerator to the known numerator of \(F(s)\):
\[
F(s) = \frac{3(s^2 + 2s + 5) + K_2(s+1)(s+2) + K_3 \cdot 2(s+2)}{(s+2)(s^2 + 2s + 5)} = \frac{9s^2 + 35s + 49}{(s+2)(s^2 + 2s + 5)}
\]
By matching the coefficients of the $s^2$ terms, we have that $K_2 = 6$. Using this value and matching the coefficients of the $s$ terms, we have the following:

$$3(2s) + 6(3s) + K_3(2s) = 35s$$

or

$$K_2 = \frac{11}{2}$$

As a check, we verify that the constant terms match:

$$3(5) + 6(2) + \frac{11}{2}(4) = 15 + 12 + 22 = 49 \quad \checkmark$$

This approach is clearly more efficient.