Ex:  

![Circuit Diagram]

a) Determine the transfer function $V_o/V_i$. **Hint:** use a Thevenin equivalent on the left side.

b) Plot $|V_o/V_i|$ versus $\omega$.

c) Find the cutoff frequency, $\omega_c$.

**Sol'n:**  
a) The technique employed in this solution focuses on writing the transfer function, $H(j \omega)$, as a product of a real-valued scaling factor and a simple RLC circuit whose transfer function, $H_1(j \omega)$, involves only one $L$, one $C$, and one $R$.

$$H(j \omega) = k \cdot H_1(j \omega) \quad \text{where } k = \text{real number}$$

The virtue of this approach is that cutoff frequencies are unaffected when a transfer function is scaled by a real number. Since the magnitude of the transfer function at the cutoff frequency is defined relative to the maximum magnitude of the transfer function, and the magnitudes are scaled by the same real number at every frequency, the cutoff frequency is unaffected by a real scaling factor multiplying a transfer function. Thus, the cutoff frequency of $H(j \omega)$ is the same as the cutoff frequency of transfer function $H_1(j \omega)$. This idea will be exploited in part (c), but it also helps to clarify the derivation of the transfer function.

Our first step is to take a Thevenin equivalent of the input voltage, $V_i$, and $R_1$ and $R_2$.
We find $V_{Th}$ as the voltage across $R_2$ when nothing is connected across $R_2$. (In other words, we remove $L$ and $R_3$ when determining the Thevenin equivalent, and $V_{Th}$ is the open-circuit voltage across $R_2$.) We have a simple voltage-divider.

$$V_{Th} = V_i \frac{R_2}{R_1 + R_2}$$

or, if we wish to write this as a transfer function, we observe that the transfer function is a real-valued scalar in terms of $R$ values:

$$H_{Th}(j\omega) = \frac{V_{Th}}{V_i} = \frac{R_2}{R_1 + R_2}$$

We find $R_{Th}$ by turning off $V_i$, (which means we replace the voltage source with a wire), and we look in from the terminals across $R_2$. We see $R_1$ and $R_2$ in parallel.

$$R_{Th} = R_1 \parallel R_2$$

At this point, we have transformed the circuit into a filter with two $R$'s and one $L$ plus a real-valued scalar multiplying the input voltage.

$$V_{Th} = V_i \frac{R_2}{R_1 + R_2}$$

Our second step is to exchange the positions of the $R_{Th}$ and $L$ so that the $R$'s are next to each other. This leaves the transfer function unaltered and allows us to consider an output signal taken across both $R$'s.
If we consider the transfer function from $V_{Th}$ to $V_1$, we have a simple $RL$ filter. We use a voltage-divider to determine the transfer function of the filter from $V_{Th}$ to $V_1$:

$$H_1(j\omega) = \frac{V_1}{V_{Th}} = \frac{R_{eq}}{R_{eq} + j\omega L}$$

where $R_{eq} = R_{Th} + R_3 = R_1 || R_2 + R_3$

Our third step is to use the voltage-divider formula to write $V_o$ as a real-valued scalar times $V_1$.

$$V_o = V_1 \frac{R_3}{R_3 + R_{Th}} = V_1 \frac{R_3}{R_{eq}} = V_1 \frac{R_3}{R_3 + R_1 || R_2}$$

or, if we wish to write this as a transfer function, we observe that the transfer function is a real-valued scalar in terms of $R$ values:

$$H_2(j\omega) = \frac{V_o}{V_1} = \frac{R_3}{R_3 + R_{Th}} = \frac{R_3}{R_3 + R_1 || R_2}$$

Our fourth step is to combine all of the above results to obtain the overall transfer function:

$$H(j\omega) = \frac{V_o}{V_i} = H_{Th}(j\omega) H_1(j\omega) H_2(j\omega) = \frac{V_{Th}}{V_i} \frac{V_1}{V_{Th}} \frac{V_o}{V_1}$$

or
Using numerical values given in the problem, we have the following results:

\[
\frac{R_2}{R_1 + R_2} = \frac{30k}{30k + 20k} = \frac{3}{5}
\]

\[
R_{eq} = 20 \text{k}\Omega \| 30 \text{k}\Omega + 3 \text{k}\Omega = 15 \text{k}\Omega
\]

\[
\frac{R_3}{R_{eq}} = \frac{3k}{15k} = \frac{1}{5}
\]

\[
\frac{R_{eq}}{R_{eq} + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R_{eq}}} = \frac{1}{1 + j\omega \frac{1}{30 \text{ M/s}}}
\]

and

\[
H(j\omega) = \frac{R_2}{R_1 + R_2} \frac{R_{eq}}{R_{eq} + j\omega L}
\]

\[
H(j\omega) = \frac{3}{25} \frac{1}{1 + j\omega \frac{1}{30 \text{ M/s}}}
\]

b) For the plot, we compute the numerical value of \(L/R_{eq}\):

\[
\frac{L}{R_{eq}} = \frac{500 \text{ } \mu \text{H}}{15 \text{ k}\Omega} = \frac{1}{30 \text{ M/s}}
\]

The following Matlab code makes the plot:

```matlab
% ECE2260F09_HW3p2Matlab.m
%
% Plot of filter's frequency response curve

R1 = 20e3;
R2 = 30e3;
R3 = 3e3;
RThev = R1 * R2 / (R1 + R2);
Req = RThev + R3;
L = 500e-6;

scaling_factor = R2/(R1 + R2) * R3/Req;
```
\[ \omega = 0:1\text{e6}:200\text{e6}; \]
\[ s = j * \omega; \]
\[ \text{FilterResp} = \text{scaling}_\text{factor} * 1./(1 + j * (L/\text{Req}) * \omega); \]
\[ \text{plot}(\omega, \text{abs}(	ext{FilterResp})) \]
\[ \text{axis}([0, \text{max}(\omega), 0, \text{scaling}_\text{factor} * 1.2]) \]
\[ \text{xlabel}(\text{\textit{omega}}) \]
\[ \text{ylabel}(\text{|H|}) \]
\[ \text{title}(\text{\textit{HW 3 prob 2 Frequency Response}}) \]

\( c) \) The cutoff frequency, \( \omega_c \), occurs where the magnitude of \( H(j\omega) \) is \( 1/\sqrt{2} \) times the maximum value attained by \( H(j\omega) \) for any \( \omega > 0 \).

The cutoff frequency of this filter, however, is the same as that of the simpler \( RL \) filter formed by the total equivalent resistance, \( R_{\text{eq}} \), and \( L \).

The transfer function of this filter, found in part (a), is as follows:

\[ H_2(j\omega) = \frac{V_2}{V_{\text{Th}}} = \frac{R_{\text{eq}}}{R_{\text{eq}} + j\omega L} \]

where \( R_{\text{eq}} = R_{\text{Th}} + R_3 = R_1 || R_2 + R_3 \)

The following Matlab code makes the plot:

\% ECE2260F09_HW3p2Matlab.m
\%
\% Plot of filter's frequency response curve
R1 = 20e3;  
R2 = 30e3;  
R3 = 3e3;  
RThev = R1 * R2 / (R1 + R2);  
Req = RThev + R3;  
L = 500e-6;  
scaling_factor = R1/(R1 + R2) * R3/Req;

figure(1)  
omega = [0:1e6:200e6];  
s = j * omega;

figure(2)  
RL_resp = 1./(1 + j * (L/Req) * omega);

plot(omega,abs(RL_resp))  
axis([0, max(omega), 0, 1.2])  
xlabel('omega')  
ylabel('|H|')  
title('HW 3 prob 2 Req L Frequency Response')

If we divide the transfer function top and bottom by $R_{eq}$, we obtain a form that allows to easily locate the cutoff frequency.

$$H_2(j\omega) = \frac{V_2}{V_{Th}} = \frac{1}{1 + j\frac{\omega L}{R_{eq}}}$$
From the plot above, the maximum magnitude of $H_2(j\omega)$ occurs when $\omega = 0$ and has a value equal to one. Thus, the cutoff frequency will occur where $H_2(j\omega) = 1/\sqrt{2}$.

$$|H_2(j\omega_c)| = \frac{1}{\sqrt{2}} \max_\omega |H_2(j\omega)| = \frac{1}{\sqrt{2}}$$

We observe that $H_2(j\omega)$ is $1/\sqrt{2}$ when the imaginary part of the denominator is equal to one.

$$|H_2(j\omega)| = \frac{1}{\sqrt{2}} \text{ when } H_2(j\omega) = \frac{1}{1 + j}$$

This holds because $|1 + j| = \sqrt{2}$.

We conclude that the cutoff frequency is found by solving the following equation:

$$\omega_c \frac{L}{R_{eq}} = 1$$

or

$$\omega_c \frac{R_{eq}}{L} = \frac{15 \text{ k}}{500 \mu \text{ s}} = 30 \text{ M r/s}$$