Ex: Find the Laplace transform of the following waveform:
\[ \int_0^t \frac{\sin(5t)}{t} \, dt \]
 Hint: \[ \frac{d}{dx} \tan^{-1}(x) = \frac{1}{x^2 + 1} \]

SOL'N: We start on the inside (of this layered onion) and apply identities to work our way out to the time-domain form given. The innermost term, found in a transform table, is \( \sin(5t) \).

\[ \mathcal{L}\{\sin(5t)\} = \frac{5}{s^2 + 5^2} \]

Now we apply the identity for multiplication by \( 1/t \).

\[ \mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty F(u) \, du \]

or

\[ \mathcal{L}\left\{ \frac{\sin(5t)}{t} \right\} = \int_s^\infty \frac{5}{u^2 + 5^2} \, du \]

Using the hint, we find the integral in a few steps.

\[ \frac{d}{dx} \tan^{-1}(x/a) = \frac{1}{(x/a)^2 + 1} \cdot \frac{1}{a} = \frac{a}{x^2 + a^2} \]

Using this result, we find the integral in the \( s \)-domain.

\[ \int_s^\infty \frac{5}{u^2 + 5^2} \, du = \tan^{-1}(u/5) \bigg|_{u=s}^{u=\infty} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{5}\right) \]

Now we apply the identity for integration in the time-domain:

\[ \mathcal{L}\left\{ \int_0^t f(\tau) \, d\tau \right\} = \frac{F(s)}{s} \]

This implies that we need only divide our previous Laplace-domain result by \( s \).

\[ \mathcal{L}\left\{ \int_0^t \frac{\sin(5t)}{t} \, dt \right\} = \frac{1}{s} \left[ \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{5}\right) \right] \]