Ex:

After being closed for a long time, the switch opens at $t = 0$.

The inductance in the above circuit acts as the coil (magnet) for a relay. The relay is designed to stay energized for a short time after the switch is opened. The relay's current waveform is designed to be over-damped. To obtain this over-damped solution, one of the characteristic roots (in units of 1/seconds) is set equal to minus the value of $R_2$ (in units of ohms).

$$s_1 = -R_2$$

The other characteristic root is set equal to minus the value of three times $R_2$:

$$s_2 = -3R_2$$

a) Find the value of $R_2$ for the circuit, given the above specifications.

b) Using the component values from Problem 1, find a numerical expression for the (downward) inductor current, $i(t)$, for $t > 0$. 
Characteristic roots are

\[ \sigma_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

where \( \alpha = \frac{R_2}{2L} \) for series RLC (for \( t > 0 \))

\[ \omega_0^2 = \frac{1}{LC} \]

Here, \( \omega_o^2 = \frac{1}{\frac{1}{4} \cdot 1.2 \cdot 1.2} = \frac{4k (r/s)^2}{1.2} \)

\[ \omega_0^2 = \frac{10k (r/s)^2}{3} \]

If \( \sigma_1 = -R_2 \) and \( \sigma_2 = -3R_2 \), then

\[ \sigma_1 + \sigma_2 = -4R_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2\alpha \]

or \( -4R_2 = -2 \frac{R_2}{2L} = -\frac{R_2}{1/4 \cdot 1.2} \)

Also, \( \sigma_1 - \sigma_2 = 2\sqrt{\alpha^2 - \omega_0^2} = -R_2 -(3R_2) \)

or \( 2\sqrt{\alpha^2 - \omega_0^2} = 2R_2 \)

or \( \alpha^2 - \omega_0^2 = R_2^2 \)

or \( \omega_0^2 = \alpha^2 - R_2^2 \)

But \( \alpha^2 = \frac{(R_2^2)}{(2L)} \cdot \frac{(R_2 - \frac{1}{4})}{(2 \cdot 1.2)} = 4 \frac{R_2^2}{3} \)

Thus \( \omega_0^2 = 4 \frac{R_2^2}{3} - R_2^2 = 3 \frac{R_2^2}{3} \)

\[ R_2 = \frac{\omega_0^2}{3} = \frac{10k}{3 \cdot 3} \Rightarrow R_2 = \frac{100 \ \Omega}{3} \]
b) Find initial conditions from $t=0^-$.  

\[ i_L(0^-) = V_0 \]

$L$ acts like wire, $C$ acts like open, switch is closed.

$L$ creates short, so we have $0V$ across $R_2$ and $C$. Thus

\[ v_c(0^-) = 0 \]

From loop on left side that carries all of the current, we have

\[ i_L(0^-) = \frac{V_0}{R_1} = \frac{30V}{150\Omega} = \frac{1}{5} A. \]

We have two real roots, so we use underdamped solution.

\[ i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 \]

\[ i(0^+) = A_1 + A_2 + A_3 \]

\[ \left. \frac{di}{dt} \right|_{t=0^+} = A_1 s_1 + A_2 s_2 \]
Since there is no power source in the circuit as $t \to \infty$, we have

$$A_3 = i(t \to \infty) = 0 \text{ A}.$$ 

We use the circuit model for $t = 0^+$ to find $i(0^+)$ and $\frac{di}{dt} \bigg|_{t=0^+}$:

$$i_L(0^+) = \frac{i_L(0^-)}{5} = \frac{1}{5} A$$

We have $i(0^+) = i_L(0^+) = \frac{1}{5} \text{ A}$.

We observe that $i(t) = i_L(t) \neq 0$

$$\frac{di}{dt} \bigg|_{t=0^+} = \frac{di_L}{dt} \bigg|_{t=0^+} = \frac{V_L(0^+)}{L}$$

From outside v-loop, we have

$$V_L(0^+) = V_C(0^+) - i_L(0^+) R_2$$

or

$$V_L(0^+) = 0V - \frac{1}{5} A \cdot \frac{100 \Omega}{3} = -\frac{20}{3} V$$

Thus,

$$\frac{di}{dt} \bigg|_{t=0^+} = -\frac{20}{3} V \cdot -\frac{1}{4} \text{ H} = \frac{80}{3} \text{ A/s}$$

So

$$A_1 + A_2 = i(0^+) = \frac{1}{5} \text{ A}$$

and

$$A_1(-R_2) + A_2(-3R_2) = -\frac{80}{3} \text{ A/s}$$