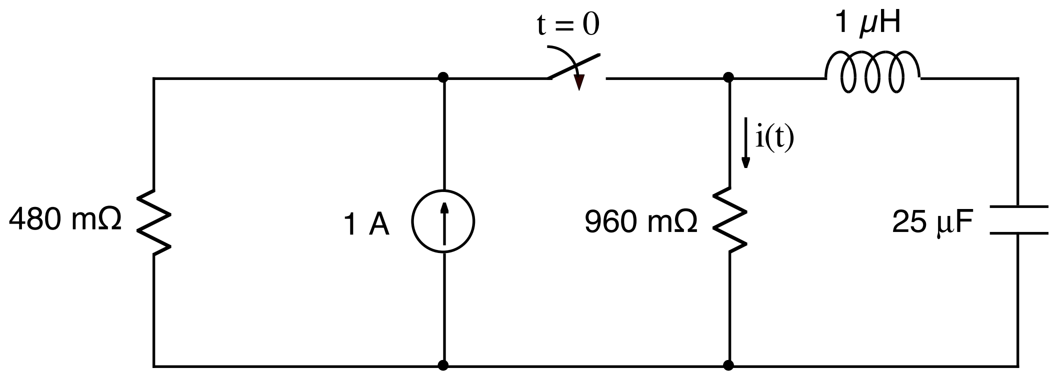
Ex:



After being open for a long time, the switch closes at $t = 0$.

Find $i(t)$ for $t > 0$.

SOL'N: We calculate characteristic roots using the circuit for $t > 0$. We set the source to zero to find R_{Thev} for the roots, which will be the parallel value of the two resistors:

$$480\ \text{m}\Omega \parallel 960\ \text{m}\Omega = 480\ \text{m}\Omega \cdot 1 \parallel 2 = 480\ \text{m}\Omega \cdot \frac{2}{3} = 320\ \text{m}\Omega$$

As for all RLC circuits, we have the following formula for the characteristic roots:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

For a series RLC circuit, the value of α is one-half the inverse L/R time constant:

$$\alpha = \frac{R}{2L} = \frac{320\ \text{m}\Omega}{2 \cdot 1\ \mu\text{H}} = 160\ \text{k/s}$$

The resonant frequency, ω_o, is the inverse of the square root of the product of L and C:

$$\omega_o = \frac{1}{\sqrt{LC}} \quad \text{or} \quad \omega_o^2 = \frac{1}{LC} = \frac{1}{1\ \mu\text{H} \cdot 25\ \mu\text{F}} = \left(\frac{1}{5\ \mu}\ \text{r/s}\right)^2 = (200\ \text{kr/s})^2$$

We find that, since $\alpha < \omega_o$, the roots are complex:

$$s_{1,2} = -160\ \text{kr/s} \pm \sqrt{(160\ \text{kr/s})^2 - (200\ \text{kr/s})^2} = -160\ \text{kr/s} \pm j120\ \text{kr/s}$$

Because the roots are complex, the circuit is under-damped:

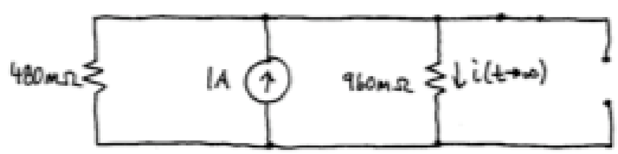
$$\omega_d = \sqrt{\omega_o^2 - \alpha^2} = \sqrt{(200\text{k})^2 - (160\text{k})^2}\ \text{r/s} = 120\ \text{kr/s}$$

We use the general form of solution for an under-damped circuit:

$$i(t) = A_1 e^{-\alpha t}\cos(\omega_d t) + A_2 e^{-\alpha t}\sin(\omega_d t) + A_3$$

A_3 = final value

For $t \to \infty$, L = wire, C = open, switch closed.



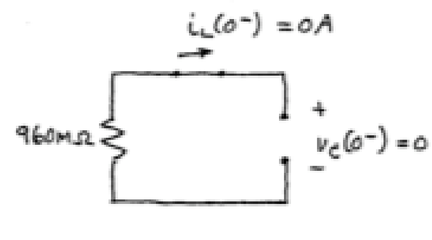
We have a current divider.

$$i(t\to\infty) = 1\text{A}\cdot\frac{480\text{m}\Omega}{480\text{m}\Omega + 960\text{m}\Omega} = \frac{1}{3}\text{A}$$

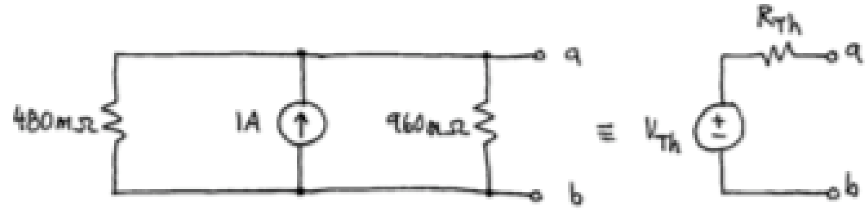
Now find $i(0^+)$ and $\left.\dfrac{di(t)}{dt}\right|_{t=0^+}$.

Start at $t=0^-$ and find $i_L(0^-)$, $v_c(0^-)$.
(Then we'll use $i_L(0^+) = i_L(0^-)$,
$v_c(0^+) = v_c(0^-)$.)

At $t=0^-$, L = wire, C = open, switch open.



For $t=0^+$, one approach is to take a Thevenin equivalent of the current source and R's.



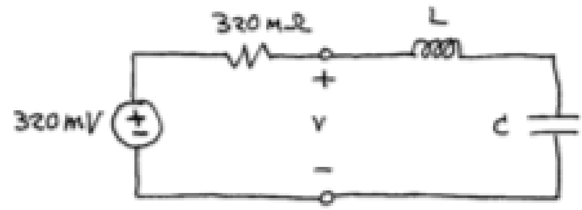
$V_{Th} = v_{a,b}$ with nothing attached to a,b.

$v_{Th} = 1A \cdot 480m\Omega \| 960m\Omega = 1A \cdot 320m\Omega = 320mV$

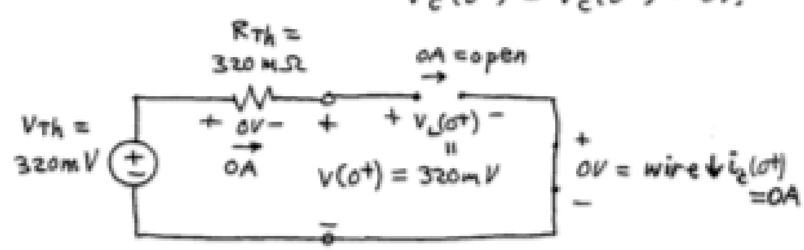
R_{Th} = resistance seen looking into a,b
with 1A source turned off

$R_{Th} = 480\,m\Omega \| 960m\Omega = 320\,m\Omega$ (as noted above)

We now find $v(t)$ in our new circuit and use $i(t) = v(t) / 960m\Omega$ from Ohm's law.



At $t=0^+$ we have $i_L(0^+) = i_L(0^-) = 0A$
$v_c(0^+) = v_c(0^-) = 0V.$



$v(0^+) = 320\,mV$ from above circuit

We match this to symbolic $v(0^+)$:

$$v(t) = A_1 e^{-\alpha t}\cos(\omega_d t) + A_2 e^{-\alpha t}\sin(\omega_d t) + A_3$$

$$v(0^+) = A_1 + A_3$$

What is A_3 for $v(t)$? It will be the A_3 we found for $i(t)$ multiplied by $960\,m\Omega$, (by Ohm's Law).

$$A_3 = \tfrac{1}{3}A \cdot 960\,m\Omega = 320\,mV \quad (= 1A \cdot R_{Th})$$

Back to $v(0^+)$, we have

$$v(0^+) = A_1 + \underset{\underset{320mV}{\shortparallel}}{A_3} = 320\,mV \text{ from circuit}$$

$\therefore A_1 = 0$

Now we find $\left.\frac{dv(t)}{dt}\right|_{t=0^+}$ by writing

$v(t)$ in terms of state vars i_L and v_C.

We must not plug in values until after we take d/dt.

$v(t) = V_{Th} - R_{Th}\, i_L$ works since i_L is state var

$$\frac{dv(t)}{dt} = \underbrace{\cancel{\frac{dV_{Th}}{dt}}}_{0 \text{ since } V_{Th} = \text{const}} - R_{Th}\frac{di_L}{dt}$$

Now use $\frac{di_L}{dt} = \frac{v_L}{L}$, $\left(\text{and } \frac{dv_C}{dt} = \frac{i_C}{C} \text{ usually}\right)$.

$$\frac{dv(t)}{dt} = -R_{Th} \frac{v_L}{L}$$

$$\left.\frac{dv(t)}{dt}\right|_{t=0^+} = -\frac{R_{Th}}{L} v_L(0^+) = -\frac{320\,m\Omega}{1\,\mu H} \cdot 320\,mV$$

From symbolic $v(t)$ we have

$$\left.\frac{dv(t)}{dt}\right|_{t=0^+} = A_1(-\alpha) + A_2\omega_d$$

Thus, $A_1(-\alpha) + A_2\omega_d = -\frac{320}{1\,\mu H} m\Omega \cdot 320\,mV$

But $A_1 = 0$. $\therefore\ A_2 = -\frac{320\ m\Omega \cdot 320\,mV}{1\,\mu H \cdot (\omega_d = 120kr/s)}$

$$v(t) = -\frac{320\,m\Omega \cdot 320\,mV}{1\,\mu H \cdot 120k\,r/s} e^{-160kt} \sin(120kt) + 320\,mV$$

$$i(t) = \frac{v(t)}{960\,m\Omega} \quad \text{since } v \text{ is across } 960\,m\Omega$$

$$i(t) = -\frac{320\,\cancel{\mu}V \cdot 1/3}{1\,\cancel{\mu}H \cdot 120\cancel{k}\,r/s} e^{-160kt} \sin(120kt) + \frac{1}{3} A$$

$$i(t) = -\frac{8A}{9} e^{-160kt} \sin(120kt) + \frac{1}{3} A$$