2. (50 points)

\[ v_1(t) = a_v + \sum_{n=1}^{\infty} A_n \cos \left( n\omega_o t + \theta_n \right) \]

Note any symmetry properties of the waveform that you use to determine coefficients.

b. The circuit on the left is a filter with output \( v_o(t) \). Design a circuit to be placed in the box such that the filter rejects the fundamental frequency of \( v_i(t) \) and has a bandwidth of 10,000 rad/sec. Specify the component values. Show how the components are connected in the circuit.
ans: a) \[ a_v = 0 \]
\[ a_n = \begin{cases} 
40 \sin \frac{\pi n}{4} & \text{n odd} \\
0 & \text{n even} 
\end{cases} \]
\[ b_n = 0 \text{ for all } n \]
\[ A_1 = \frac{20\sqrt{2}}{\pi}, \quad \theta_1 = 0^\circ \
A_2 = 0, \quad \theta_2 = 0^\circ 
A_3 = \frac{20\sqrt{2}}{3\pi}, \quad \theta_3 = 0^\circ 
\]
\[ A_4 = 0, \quad \theta_4 = 0^\circ 
A_5 = \frac{-4\sqrt{2}}{\pi}, \quad \theta_5 = 0^\circ \]

Symmetries used: even function, half wave (shift-flip symmetry), and quarter wave symmetry.

b)

sol'n: (a) \[ a_v = \text{ave value of } v_i(t) = 0 \text{ since equal positive and negative areas are under the } v_i(t) \text{ curve.} \]
\[ v_i(t) \text{ is symmetric around vertical axis so } v_i(t) \text{ is an even function. This} \]
\[ \text{means we need only even functions—cosine terms—in our Fourier series.} \]
\[ \therefore b_n = 0 \text{ for all } n \text{ (no } \sin (n\omega_0 t) \text{ terms in Fourier series)} \]

If we shift \( v_i(t) \) one-half period and flip it upside down, we have \( v_i(t) \) again. Thus, we have half-wave symmetry or, as refer to it, shift-flip symmetry.
\[ \therefore a_n = 0 \text{ for } n \text{ even (} b_n = 0 \text{ for } n \text{ even, too, but we already know } b_n = 0 \text{ all } n) \]

For the question of quarter wave symmetry, we look for symmetry around \( T/4 \) and \( 3T/4 \). What we find is that \( v_i(t) \) is odd around \( T/4 \) and \( 3T/4 \). In other words, if the vertical axis for \( T = 0 \) were shifted to \( T/4 \) or \( 3T/4 \), \( v_i(t) \) would be an odd function. If we superimpose the \( \cos(n\omega_0 t) \) term for \( n = 1 \) on \( v_i(t) \) and consider the signs of the product \( v_i(t)\cos(n\omega_0 t) \), as shown below, we discover that we can calculate \( a_1 \) by quadrupling the integral from 0 to \( T/4 \) in the formula for \( a_1 \):
\[ a_1 = 4 \cdot \frac{2}{T} \int_0^{T/4} v_i(t) \cos(1 \cdot \omega_0 t) \, dt \]

The same will hold true for every odd numbered \( n \).

Now we define \( v_i(t) \) from 0 to \( T/4 \):

\[
v_i(t) = \begin{cases} 
10 & 0 \leq t \leq T/8 \\
0 & T/8 < t \leq T/4
\end{cases}
\]

Thus,

\[
a_n = \frac{8}{T} \left[ \int_0^{T/8} 10 \cos(n \omega_0 t) \, dt + \int_0^{T/8} 0 \cdot \cos(n \omega_0 t) \, dt \right]
\]

or

\[
a_n = \frac{8}{T} \int_0^{T/8} 10 \cos(n \omega_0 t) \, dt
\]

\[
= \frac{8}{T} \left. 10 \sin(n \omega_0 t) \right|_0^{T/8}
\]

Now substitute:

\[
\omega_0 = \frac{2\pi}{T}
\]
\[ a_n = \frac{8}{\pi} \cdot 10 \sin \frac{n \pi}{T} \left( \frac{T/8}{2 \pi} \right) \]

\[ = \frac{40}{\pi n} \sin \frac{2\pi n \pi}{T} - \sin \frac{0}{0} \]

\[ a_n = \frac{40}{\pi n} \sin \left( \frac{\pi n}{4} \right) \quad \text{for } n \text{ odd} \]

If we compute the values of \( \sin(\pi n/4) \) for \( n = 0, 1, \ldots \) we get 0, \( 1/\sqrt{2} \), 1, \( 1/\sqrt{2} \), 0, \( -1/\sqrt{2} \), \( -1/\sqrt{2} \), 0, in a repeating pattern.

Therefore, \( a_n \) coefficients for \( n \) odd up to the fifth harmonic are:

\[ a_1 = \sqrt{2} \cdot \frac{40}{2 \pi}, \quad a_3 = \sqrt{2} \cdot \frac{40}{3\pi}, \quad a_5 = -\sqrt{2} \cdot \frac{40}{5\pi} \]

Now we convert to phasor form, \( a_n \cos (\omega_0 t) + b_n \sin (\omega_0 t) \). The time-domain rectangular representation of the \( n \)th term of the Fourier series is

\[ a_n \cos (\omega_0 t) + b_n \sin (\omega_0 t) \]

Recalling that the phasor for pure \( \cos() \) is 1 and for pure \( \sin() \) is \(-j\), the phasor for the \( n \)th term of the Fourier series is

\[ a_n \text{ (or } a_n \angle 0^\circ) + -j b_n \text{ (or } b_n \angle -90^\circ) \]

Thus, our phasor is \( a_n - j b_n \). Incidentally, if we convert to polar form, \( A_n \angle \theta_n \), we have:

\[ A_n = \sqrt{a_n^2 + b_n^2} \]

\[ \theta_n = \tan^{-1} \left( \frac{-b_n}{a_n} \right) \]

Here, however, all \( b_n = 0 \). So we have \( A_n = a_n, \theta_n = 0^\circ \). In other words, we have only \( \cos() \) terms, and the phase angle for \( \cos() \) terms is zero since they are real.

\[ A_1 = a_1 = \frac{20\sqrt{2}}{\pi}, \quad \theta_1 = 0^\circ \]

\[ A_3 = a_3 = \frac{20\sqrt{2}}{3\pi}, \quad \theta_3 = 0^\circ \]
\[ A_5 = a_5 = \frac{-20\sqrt{2}}{5\pi}, \quad \theta_5 = 0^\circ \]

**Note:** You may find it easier to derive symmetry results by drawing \( v_i(t) \) and the \( \cos(\cdot) \) or \( \sin(\cdot) \) waveforms on a plot and multiplying them point by point (a rough sketch will do). The area under the curve corresponds to

\[
\int_0^T v_i(t) \cos(\cdot) \quad \text{or} \quad \int_0^T v_i(t) \sin(\cdot)
\]

If the positive and negative areas under the product curves cancel, \( a_n \) (or \( b_n \)) = 0.

**sol'n: (b)** We want a band reject filter with center frequency = \( \omega_o = 1M \text{ rad/s} \), (see diagram in problem statement), and bandwidth \( \beta = 10k \text{ rad/s} \) (see problem statement).

Note: By coincidence, in this problem \( \omega_o \) for the Fourier series (which is determined by the value of the period, \( T \)), happens to be the same as the center frequency, \( \omega_o \), of the filter (which is determined the values of \( R \), \( L \), and \( C \)). This need not always be the case.

Our transfer function is \( H(s) \equiv \frac{V_o(s)}{V_i(s)} \).

We use V-divider formula for \( V_o(s) \) in terms of \( V_i(s) \), letting \( z_L \) denote the impedance in the box.

\[
V_o(s) = V_i(s) \cdot \frac{z_L}{1 k\Omega + z_L}
\]

\[
H(s) = \frac{V_o(s)}{V_i(s)} = \frac{z_L}{1 k\Omega + z_L}
\]

We need \( z_L = 0 \) at \( \omega = 1M \) to get

\[
\frac{V_o(s = j\omega = j1Mr/s)}{V_i(s = j\omega = j1Mr/s)} = 0
\]

We use an L in series with a C to get \( z \) cancellation:
To get cancellation, $\omega L = 1/\omega C$ at $\omega = 1$M or

$$LC = \frac{1}{\omega^2} = \frac{1}{(1\text{M})^2} = 1 \text{ ps}$$

We have RLC in series, and for a series RLC band-reject filter, we have $\beta = R/L$. For $\beta = 10$ krad/s and $R = 1$ k$\Omega$, we get

$L = \frac{R}{\beta} = 0.1$ H.

Knowing $L$, we can now solve for $C$:

$$C = \frac{1}{L\omega^2} = \frac{1}{0.1\text{H}(1\text{M/s})^2}$$

$\therefore \quad C = 10 \text{ pF}$