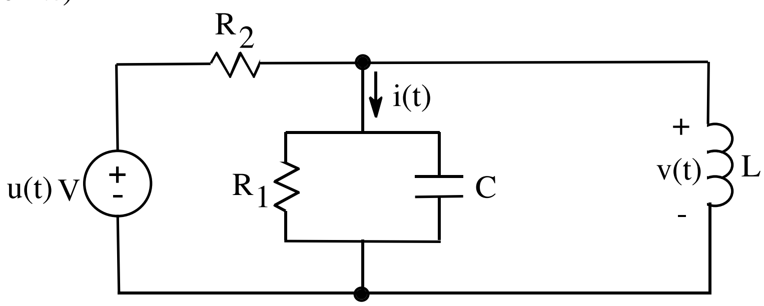
3. (50 points)



The initial energy stored in the circuit is zero.

$R_2 = 500\ \Omega$ $L = 200$ mH

a. Choose values of R_1 and C to accomplish the following:

(1) v(t) and i(t) are decaying sinusoids 90° out of phase with each other.

(2) $1/\alpha = T$, where α is the exponential decay constant and T is the period of oscillation of the decaying sinusoid.

b. With the component values you chose in the circuit, write numerical expressions for v(t) and i(t).

ans: a) $R_1 = 500\Omega$, $C = 32.4\ \mu F$

b) $v(t) = 159\ e^{-61.7t} \sin(388t)$ mV

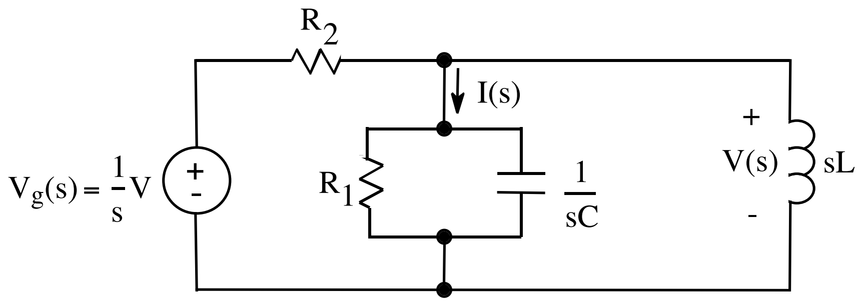
$i(t) = 2\ e^{-61.7t} \cos(388t)$ mA

sol'n: (a) This problem could actually be solved by solving the RLC problem with the general solution to the differential equation for a parallel RLC (obtained by taking the Thevenin equivalent of the voltage source and two resistors). The Laplace transform approach is a bit more straightforward, however, and would also work for input sources other than a step function.

We begin by Laplace transforming the source and circuit components. Since no initial energy is stored in the circuit, we have no extra current or voltage sources in the component models. In general, we would have to include such sources to account for the initial conditions.

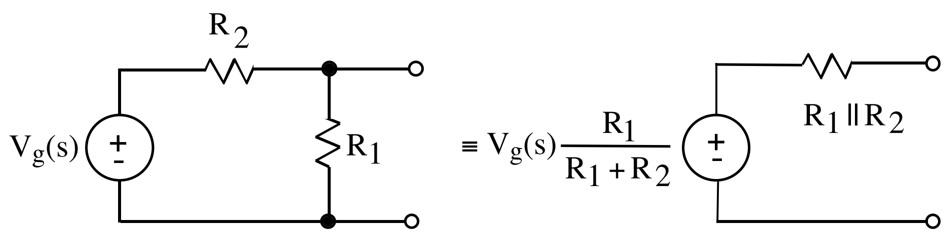
$$\mathcal{L}\{u(t)\} = \frac{1}{s} \quad R \xrightarrow{\mathcal{L}} R \quad L \xrightarrow{\mathcal{L}} s\mathrm{L} \quad C \xrightarrow{\mathcal{L}} \frac{1}{sC}$$

s-domain model:



First, we find V(s). We can then find I(s) from V(s) using Ohm's law.

Our calculations are cleaner if we use a Thevenin equivalent for $V_g(s)$, R_1, and R_2.

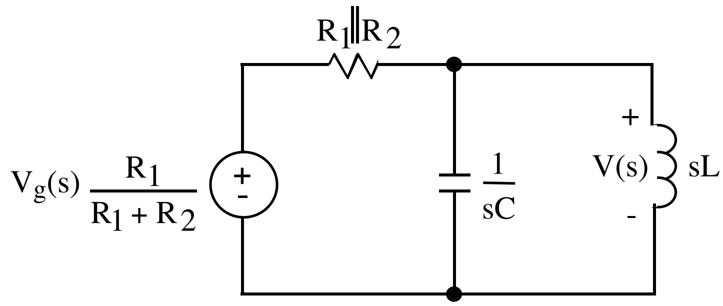


Note: After we replace $V_g(s)$, R_1, and R_2 with the Thevenin equivalent, we no longer have a point in the circuit where I(s) is flowing. Nevertheless, we may still compute I(s) from V(s):

$$I(s) = \frac{V(s)}{R_1 \,||\, \frac{1}{sC}}$$

(Note that V is across R_1 and 1/sC as well as across sL.)

Our circuit model becomes:



The impedance of the L and C in parallel is

$$\frac{1}{sC} \,||\, sL = \frac{\frac{sL}{sC}}{\frac{1}{sC} + sL} = \frac{sL}{1 + s^2 LC} = \frac{\frac{s}{C}}{s^2 + \frac{1}{LC}}$$

Use V-divider to find V:

$$V(s) = V_g(s)\frac{R_1}{R_1 + R_2} \cdot \frac{\frac{1}{sC} \parallel sL}{\frac{1}{sC} \parallel sL + R_1 \parallel R_2}$$

Now simplify the expression.

$$V(s) = V_g(s)\frac{R_1}{R_1 + R_2} \cdot \frac{\dfrac{\frac{s}{C}}{s^2 + \frac{1}{LC}}}{\dfrac{\frac{s}{C}}{s^2 + \frac{1}{LC}} + R_1 \| R_2}$$

Multiply top and bottom by $s^2 + 1/LC$:

$$V(s) = V_g(s)\frac{R_1}{R_1 + R_2} \cdot \frac{\frac{s}{C}}{\frac{s}{C} + \left(s^2 + \frac{1}{LC}\right) R_1 \parallel R_2}$$

Rewrite the voltage divider as $R_1 \| R_2 / R_2$:

$$V(s) = V_g(s)\frac{R_1 \parallel R_2}{R_2} \cdot \frac{\frac{s}{C}}{\frac{s}{C} + \left(s^2 + \frac{1}{LC}\right) R_1 \parallel R_2}$$

Divide top and bottom by $R_1 \| R_2$:

$$V(s) = V_g(s)\frac{\frac{s}{R_2 C}}{\frac{s}{R_1 \parallel R_2 C} + \left(s^2 + \frac{1}{LC}\right)}$$

Tidy up the denominator and substitute $V_g = 1/s$:

$$V(s) = \frac{1}{s}\frac{\frac{s}{R_2 C}}{s^2 + \frac{s}{R_1 \parallel R_2 C} + \frac{1}{LC}} = \frac{\frac{1}{R_2 C}}{s^2 + \frac{s}{R_1 \parallel R_2 C} + \frac{1}{LC}}$$

V has the form of a decaying sine wave:

$$\mathcal{L}\left\{ke^{-\alpha t}\sin(\omega t)\right\} = \frac{k\omega}{(s+\alpha)^2+\omega^2} = \frac{k\omega}{s^2+2\alpha s+\alpha^2+\omega^2}$$

where k is a real constant and

$$k\omega = \frac{1}{R_2 C}, \quad \alpha^2+\omega^2 = \frac{1}{LC}, \quad \text{and } 2\alpha = \frac{1}{\mathrm{R}_1 \,||\, \mathrm{R}_2 C}.$$

ASIDE: The fundamental form of the polynomials appearing in V(s) is independent of the value of R_1, R_2, L, or C. If I(s) is to be 90° out of phase with V(s), we must hope that it can be made to have the form of a decaying cosine:

$$\mathcal{L}\left\{k_2 e^{-\alpha t}\cos(\omega t)\right\} = \frac{k_2(s+\alpha)}{(s+\alpha)^2+\omega^2} = \frac{k_2(s+\alpha)}{s^2+2\alpha s+\alpha^2+\omega^2}$$

where k_2 is a real constant (positive or negative).

I(s) is V(s) divided by the parallel impedance of R_1 and C:

$$I(s) = \frac{V(s)}{R_1 \,||\, \dfrac{1}{sC}}$$

The parallel impedance of R_1 and C is

$$R_1 \left\| \frac{1}{sC} \right. = \frac{\dfrac{R_1}{sC}}{R_1 + \dfrac{1}{sC}} = \frac{R_1}{1+sR_1C}.$$

Substituting for V(s) gives

$$I(s) = \frac{\dfrac{\dfrac{1}{R_2C}}{s^2 + \dfrac{s}{R_1 \,||\, R_2C} + \dfrac{1}{LC}}}{\dfrac{R_1}{1+sR_1C}} = \frac{\dfrac{1}{R_2C}}{s^2 + \dfrac{s}{R_1 \,||\, R_2C} + \dfrac{1}{LC}} \cdot \frac{1+sR_1C}{R_1}$$

$$I(s) = \frac{\dfrac{1+sR_1C}{R_1R_2C}}{s^2 + \dfrac{s}{R_1 \,||\, R_2C} + \dfrac{1}{LC}}$$

Above, we concluded that I(s) must be of form

$$I(s) = \frac{k_2(s+\alpha)}{s^2 + 2\alpha s + \alpha^2 + \omega^2}$$

The denominator polynomial in *s* is in the correct form. Matching the numerators gives:

$$\frac{1}{R_2}\left(s + \frac{1}{R_1 C}\right) = k_2(s+\alpha)$$

We must have $k_2 = 1/R_2$ and $\alpha = 1/R_1C$. Earlier, we found

$$\alpha = \frac{1}{2(R_1 \parallel R_2)C}.$$

We conclude that

$$R_1 = 2(R_1 \parallel R_2).$$

The solution to this equation is $R_1 = R_2 = 500\ \Omega$, (value given in problem statement):

$$R_1 = 500\Omega$$

A standard value for R_1 would be 510 Ω.

Now we find C. We look for an equation with C as the only unknown. The problem states that $1/\alpha = T$ or $\alpha = 1/T = \omega/2\pi$:

$$\alpha = \frac{1}{2(R_1 \parallel R_2)C} = \frac{1}{2\pi}\omega$$

From the constant term in denominator of V(s) and I(s), as noted earlier, we have an equation involving α and ω:

$$\alpha^2 + \omega^2 = \frac{1}{LC}$$

Substituting for α and ω, we have

$$\left[\frac{1}{2(R_1 \parallel R_2)C}\right]^2 \left[1 + (2\pi)^2\right] = \frac{1}{LC}.$$

Inverting both sides, we have

$$\left[2(R_1 \parallel R_2)C\right]^2 \frac{1}{1 + (2\pi)^2} = LC.$$

Rearranging and canceling one power of C on both sides gives a value for C:

$$C = \frac{1+(2\pi)^2}{4(R_1 \parallel R_2)^2} L$$

$$C = \frac{1+(2\pi)^2}{4(500\Omega \parallel 500\Omega)^2} 200\text{mH}$$

$$C = \frac{1+(2\pi)^2}{4 \cdot 250 \cdot 250\Omega^2} 200\text{mH} = \frac{1+(2\pi)^2}{1\text{k} \cdot 250\Omega^2} 200\text{mH}$$

$$C = \left[1+(2\pi)^2\right] \frac{200}{250} \frac{\mu\text{H}}{\Omega^2}$$

$$C = 32.4\mu\text{F}$$

A standard value for C would be 33 μF.

sol'n: (b) From (a) we know

$$v(t) = ke^{-\alpha t}\sin(\omega t) \text{ and}$$

$$i(t) = k_2 e^{-\alpha t}\cos(\omega t).$$

From (a) we also know

$$k = \frac{1}{R_2 C\omega} = \frac{1}{R_2 C \frac{2\pi}{\alpha}} = \frac{R_2 C}{R_2 C \cdot 2\pi} = \frac{1}{2\pi} \text{ and } k_2 = \frac{1}{R_2}.$$

$$k = \frac{1\text{V}}{2\pi} = 159 \text{ mV}$$

$$k_2 = \frac{1\text{V}}{500\Omega} = 2 \text{ mA}$$

Note: We use 1V in the numerators because $V_g(t)$ actually transforms to 1V/*s* rather than 1/*s*. We leave out the 1V to avoid clutter and confusion in the calculations. We might mistakenly think of the 1V as V(s).

For α and ω, we have

$$\alpha = \frac{1}{500\Omega\ 32.4\ \mu F} = 61.7/\text{s and } \omega = 2\pi\alpha = 388\text{rad/s}.$$

Substituting numerical values for k, k_2, α, and ω in the time-domain forms for i(t) and v(t) gives our final answer.

$v(t) = 159\ e^{-61.7t} \sin(388t)$ mV

$i(t) = 2\ e^{-61.7t} \cos(388t)$ mA

Consistency check:

$$i(t) = i_{R1} + i_C = \frac{v(t)}{R_1} + C\frac{dv(t)}{dt} = \frac{k}{R_1} e^{-\alpha t} \sin(\omega t) + C\frac{d}{dt}[k e^{-\alpha t} \sin(\omega t)]$$

$$= \frac{k}{R_1} e^{-\alpha t} \sin(\omega t) + Ck[(-\alpha) e^{-\alpha t} \sin(\omega t) + e^{-\alpha t} \cos(\omega t)]$$

$$= k[\frac{\cancel{1}}{\cancel{R_1}} + (-\alpha C = -\frac{\cancel{C}}{\cancel{R_1 C}})]\ e^{-\alpha t} \sin(\omega t) \ + (\cancel{C}k = \frac{1}{R_1 \cancel{C} \cancel{\omega}})\ \cancel{\omega}\ e^{-\alpha t} \cos(\omega t)$$

$$= \frac{1}{R_1} e^{-\alpha t} \cos(\omega t) = i(t)\ \checkmark \qquad\qquad (\text{since } R_1 = R_2)$$