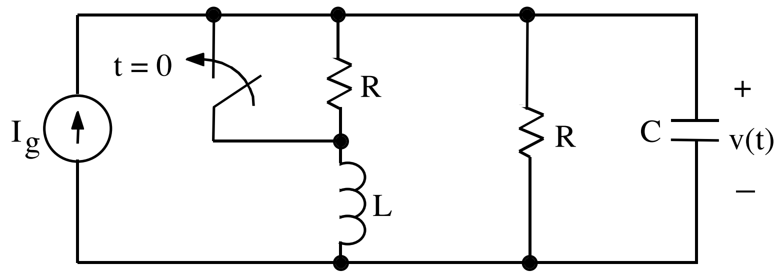
2. (45 points)




$I_g = 1$A

$R = 4\Omega$

$L = 0.1$ mH

$C = 1\mu$F

The current source is a dc current source. After being open for a long time, the switch is closed at t = 0.

a. Write a numerical time-domain expression for v(t).

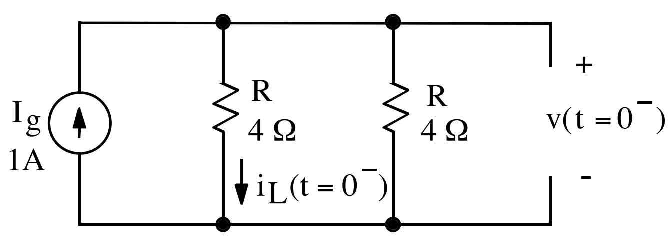
b. From the Laplace transform of v(t), find the numerical values of v(t) for $t = 0^+$ and $t \to \infty$.

**ans: a)** $v(t > 0) = \frac{8}{3} e^{-50kt} - \frac{2}{3} e^{-200kt}$ V

**b)** $v(t = 0^+) = 2$V, $v(t \to \infty) = 0$V

**sol'n: (a)** First we find initial conditions for L and C. (We need these for s-domain models of L and C.)

For $t = 0^-$, L acts like short, C acts like open circuit.

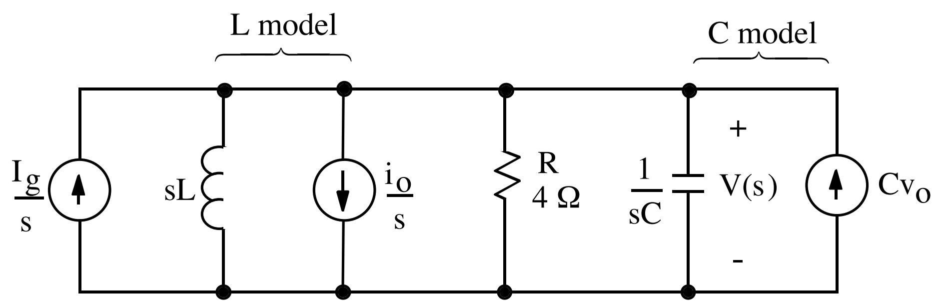



$$v(t = 0^-) = I_g \cdot R \| R = 1\text{A} \cdot 2\Omega = 2\text{V} \equiv v_o$$

$$i_L(t = 0^-) = I_g \cdot \frac{R}{R + R} = \frac{1}{2}\text{A} \equiv i_o$$

When we close the switch, we short out the first R.

s-domain model:



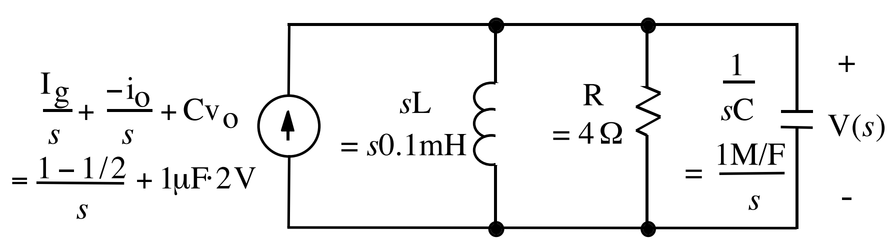

**Note:** A DC source corresponds to a step function <u>even if there is no switch and the source has the same output for all time.</u> Thus, we have $I_g/s$ as the source in the *s*-domain. (Conceptually, we only need the current source for t > 0 because the initial conditions on L and C account for what the current source did for t < 0.)

$$\mathcal{L}\{I_g\} = \mathcal{L}\{I_g u(t)\} = \frac{1}{s}$$

**Note:** We may choose either a series *s*L and V-source for L or a parallel sL and I-source for L. Here, the parallel I-source model is more convenient. The same applies to the C.

Normally, we might use superposition at this point, turning on the I-sources one at a time and then summing currents or voltages to get a final answer.

Here, however, we have parallel I-sources that sum:




Combining the parallel impedances and using **V** = **I**z, we have

$$V(s) = \left(\frac{1}{2s} + 2\mu\text{FV}\right) \cdot sL \parallel R \parallel \frac{1}{sC}$$

To compute the parallel *z* value, we factor out numerators and use the following identity:

$$\frac{1}{a}\left\|\frac{1}{b}\right\|\frac{1}{c} = \frac{1}{a+b+c}$$

Thus, we factor out $sL$ and $R$:

$$sL\|R\|\frac{1}{sC} = sLR \cdot \frac{1}{R}\Big\|\frac{1}{sL}\Big\|\frac{1}{sLRsC} = \frac{sLR}{R + sL + s^2RLC}$$

Now divide by $RLC$ to get denominator in proper form:

$$sL\|R\|\frac{1}{sC} = \frac{1}{C}\frac{s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

**Check:** Using the numerator and the first term in the denominator, we have the following units analysis:

$$\frac{s/C}{s^2} = \frac{1}{sC}$$

Thus, we have an impedance as we should have. The other terms in the denominator have the same units as $s^2$ since the units of $s$ are, ironically, 1/sec or 1/s.

Now we plug in numbers to compute V($s$):

$$V(s) = \left(\frac{1}{2s} + 2\mu\right)\left(\frac{\overset{1/C}{\overbrace{1\text{M}}}\,\not{s}}{s^2 + \underbrace{\frac{1\text{M}}{4}}_{\frac{1}{RC}}s + \underbrace{10\text{G}}_{\frac{1}{LC}}}\right)$$

$$\frac{1}{2s} + 2\mu = \frac{1 + s4\mu}{2\not{s}}$$

$\frac{1}{RC}$ and $\frac{1}{LC}$: quadratic poles term

Find poles for quadratic term in preparation for partial fractions:

$$s_{1,2} = -\frac{1\text{M}}{8} \pm \sqrt{\left(\frac{1\text{M}}{8}\right)^2 - 10\text{G}} \qquad \underline{\text{not}} \text{ complex poles}$$

$$\sqrt{\left(\frac{1\text{M}}{8}\right)^2 - 10\text{G}} = \sqrt{(125\text{k})^2 - (100\text{k})^2}$$

$s_{1,2}$ **=** -125k ± 75k rad/s (based on $5^2 - 4^2 = 3^2$ pythagorean triple)

$s_1 = –50\text{k}$, $s_2 = –200$ rad/s

Now use partial fractions:

$$V(s) = \frac{k_1}{s + 50\text{k}} + \frac{k_2}{s + 200\text{k}}$$

$$k_1 = V(s)(s+50k)\Big|_{s=-50k} = \frac{1-50k\cdot 4\mu}{2}\cdot\frac{1M}{-50k+200k}$$

$$= \frac{1-200m}{2}\,\frac{1M}{150k}$$

$$= \frac{800m}{2}\,\frac{1M}{150k} = \frac{8}{3}$$

$$k_2 = V(s)(s+50k)\Big|_{s=-200k} = \frac{1-200k4\mu}{2}\,\frac{1M}{-200k+50k}$$

$$= \frac{m\cdot 1M}{-2(150k)} = -\frac{2}{3}$$

$$V(s) = \frac{8/3}{s+50k} - \frac{2/3}{s+200k}$$

Use the standard inverse Laplace transform term:

$$\mathcal{L}^{-1}\left\{\frac{k}{s+a}\right\} = ke^{-at}$$

This gives the final answer:

$$v(t>0) = \frac{8}{3}e^{-50kt} - \frac{2}{3}e^{-200kt}\ V$$

**sol'n: (b)** Use the initial value theorem to find $v(t=0^+)$:

$$v(t=0^+) = \lim_{s\to\infty} sV(s) = \lim_{s\to\infty} s\,\frac{1+s4\mu}{2}\,\frac{1M}{s^2+\frac{1M}{4}s+10G}$$

The largest power of s dominates in the numerator and in the denominator.

$$v(t=0^+) = \lim_{s\to\infty} sV(s) = \lim_{s\to\infty}\frac{s^2 4\mu 1M}{2s^2} = \frac{4}{2} = 2V\ \surd$$

**Note:** We expect $v(0^+) = 2V$ since this is the initial capacitor voltage.

Use the final value theorem to find $v(t\to\infty)$:

$$v(t\to\infty) = \lim_{s\to 0} sV(s) = \lim_{s\to 0} s\,\frac{1+s4\mu}{2}\,\frac{1M}{s^2+\frac{1M}{4}s+10G}$$

$$= 0\cdot\frac{1}{2}\cdot\frac{1M}{10G} = 0V\ \checkmark$$

**Note:** We expect v(t) to decay, since L becomes a short.