**TOOL:** The following transform pair is useful for determining the inverse Laplace transform when coefficients and roots are complex:

\[
\mathcal{L}\left[e^{-at} (2c \cos \omega t + 2d \sin \omega t)u(t)\right] = \frac{c + jd}{s + a + j\omega} + \frac{c - jd}{s + a - j\omega}
\]

or

\[
\mathcal{L}\left[e^{-at} (2\text{Re}[A] \cos \omega t + 2\text{Im}[A] \sin \omega t)u(t)\right] = \frac{A}{s + a + j\omega} + \frac{A^*}{s + a - j\omega}
\]

If we express the sum of \(\cos()\) and \(\sin()\) as a cosine with a phase shift, we obtain the following result:

\[
\mathcal{L}\left[e^{-at} 2\sqrt{c^2 + d^2} \cos\left(\omega t + \tan^{-1} \frac{d}{c}\right)u(t)\right] = \frac{c + jd}{s + a + j\omega} + \frac{c - jd}{s + a - j\omega}
\]

or

\[
\mathcal{L}\left[e^{-at} 2|A| \cos\left(\omega t + \tan^{-1} \frac{\text{Im}[A]}{\text{Re}[A]}\right)u(t)\right] = \frac{A}{s + a + j\omega} + \frac{A^*}{s + a - j\omega}
\]

**DERIV:** We begin with a partial fraction expansion with complex coefficients and complex roots. The coefficients will be complex conjugates, and we write the coefficients as \(c + jd\) and \(c - jd\) where \(c\) and \(d\) are real.

\[
F(s) = \frac{c + jd}{s + a + j\omega} + \frac{c - jd}{s + a - j\omega}
\]

We use a common denominator to identify terms that correspond to a decaying cosine and decaying cosine, (see below):

\[
F(s) = \frac{(c + jd)(s + a - j\omega) + (c - jd)(s + a + j\omega)}{(s + a)^2 + \omega^2}
\]

A number of terms cancel in the numerator:

\[
F(s) = \frac{2c(s + a) + 2d\omega}{(s + a)^2 + \omega^2} = \frac{2c(s + a)}{(s + a)^2 + \omega^2} + \frac{2d\omega}{(s + a)^2 + \omega^2}
\]

The first term on the right corresponds to a decaying cosine waveform in the time-domain, and the second term on the right corresponds to a decaying sine waveform in the time-domain:
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LAPLACE TRANSFORM

Matching forms

EXAMPLE 1 (CONT.)

\[
\mathcal{L}\left[ e^{-at} (2c \cos \omega t + 2d \sin \omega t)u(t) \right] = \frac{2c(s + a)}{(s + a)^2 + \omega^2} + \frac{2d\omega}{(s + a)^2 + \omega^2}
\]

To express this in terms of a cosine with a phase shift, we use trigonometric identities:

\[
K \cos(\omega t + \theta) = K \cos(\omega t) \cos \theta - K \sin(\omega t) \sin \theta
\]

It follows that the following equations must hold:

\[
K \cos \theta = 2c \quad \text{and} \quad K \sin \theta = 2d
\]

Squaring both terms and summing yields the following result:

\[
K^2 = K^2 \cos^2 \theta + K^2 \sin^2 \theta = 2^2 (c^2 + d^2)
\]

or

\[
K = 2\sqrt{c^2 + d^2}
\]

Taking the ratio of the \( \cos() \) and \( \sin() \) equations yields the following result:

\[
\frac{\tan \theta}{K \cos \theta} = \frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{2c} = \frac{\tan \theta}{\cos \theta}
\]

or

\[
\theta = \tan^{-1} \frac{d}{c}
\]