APPENDIX E — Bode Diagrams

As we have seen, the frequency response plot is a very important tool for analyzing a circuit's behavior. Up to this point, however, we have shown qualitative sketches of the frequency response without discussing how to create such diagrams. The most efficient method for generating and plotting the amplitude and phase data is to use a digital computer; we can rely on it to give us accurate numerical plots of $|H(j\omega)|$ and $\theta(j\omega)$ versus ω. However, in some situations, preliminary sketches using Bode diagrams can help ensure the intelligent use of the computer.

A Bode diagram, or plot, is a graphical technique that gives a feel for the frequency response of a circuit. These diagrams are named in recognition of the pioneering work done by H. W. Bode.[1] They are most useful for circuits in which the poles and zeros of $H(s)$ are reasonably well separated.

Like the qualitative frequency response plots seen thus far, a Bode diagram consists of two separate plots: One shows how the amplitude of $H(j\omega)$ varies with frequency, and the other shows how the phase angle of $H(j\omega)$ varies with frequency. In Bode diagrams, the plots are made on semilog graph paper for greater accuracy in representing the wide range of frequency values. In both the amplitude and phase plots, the frequency is plotted on the horizontal log scale, and the amplitude and phase angle are plotted on the linear vertical scale.

E.1 ◆ Real, First-Order Poles and Zeros

To simplify the development of Bode diagrams, we begin by considering only cases where all the poles and zeros of $H(s)$ are real and first order. Later we will present cases with complex and repeated poles and zeros. For our purposes, having a specific expression for $H(s)$ is helpful. Hence we base the discussion on

$$H(s) = \frac{K(s + z_1)}{s(s + p_1)}, \tag{E.1}$$

from which

$$H(j\omega) = \frac{K(j\omega + z_1)}{j\omega(j\omega + p_1)}. \tag{E.2}$$

The first step in making Bode diagrams is to put the expression for $H(j\omega)$ in a **standard form**, which we derive simply by dividing out the

[1] See H. W. Bode, *Network Analysis and Feedback Design* (New York: Van Nostrand, 1945).

poles and zeros:

$$H(j\omega) = \frac{Kz_1(1 + j\omega/z_1)}{p_1(j\omega)(1 + j\omega/p_1)}. \tag{E.3}$$

Next we let K_o represent the constant quantity Kz_1/p_1, and at the same time we express $H(j\omega)$ in polar form:

$$H(j\omega) = \frac{K_o|1 + j\omega/z_1|\underline{/\psi_1}}{|\omega|\ \underline{/90^\circ}\ |1 + j\omega/p_1|\underline{/\beta_1}}$$

$$= \frac{K_o|1 + j\omega/z_1|}{|\omega||1 + j\omega/p_1|}\underline{/(\psi_1 - 90^\circ - \beta_1)}. \tag{E.4}$$

From Eq. E.4,

$$|H(j\omega)| = \frac{K_o|1 + j\omega/z_1|}{\omega|1 + j\omega/p_1|}, \tag{E.5}$$

$$\theta(\omega) = \psi_1 - 90^\circ - \beta_1. \tag{E.6}$$

By definition, the phase angles ψ_1 and β_1 are

$$\psi_1 = \tan^{-1} \omega/z_1; \tag{E.7}$$

$$\beta_1 = \tan^{-1} \omega/p_1. \tag{E.8}$$

The Bode diagrams consist of plotting Eq. E.5 (amplitude) and Eq. E.6 (phase) as functions of ω.

E.2 ◆ Straight-Line Amplitude Plots

The amplitude plot involves the multiplication and division of factors associated with the poles and zeros of $H(s)$. We reduce this multiplication and division to addition and subtraction by expressing the amplitude of $H(j\omega)$ in terms of a logarithmic value: the decibel (dB).[2] The amplitude of $H(j\omega)$ in decibels is

$$A_{dB} = 20 \log_{10} |H(j\omega)|. \tag{E.9}$$

To give you a feel for the unit of decibels, Table E.1 provides a translation between the actual value of several amplitudes and their values in decibels. Expressing Eq. E.5 in terms of decibels gives

$$A_{dB} = 20 \log_{10} \frac{K_o|1 + j\omega/z_1|}{\omega|1 + j\omega/p_1|}$$

$$= 20 \log_{10} K_o + 20 \log_{10} |1 + j\omega/z_1|$$

$$- 20 \log_{10} \omega - 20 \log_{10} |1 + j\omega/p_1|. \tag{E.10}$$

TABLE E.1 Actual Amplitudes and Their Decibel Values

A_{dB}	A	A_{dB}	A
0	1.00	30	31.62
3	1.41	40	100.00
6	2.00	60	10^3
10	3.16	80	10^4
15	5.62	100	10^5
20	10.00	120	10^6

[2] See Appendix D for more information regarding the decibel.

The key to plotting Eq. E.10 is to plot each term in the equation separately and then combine the separate plots graphically. The individual factors are easy to plot because they can be approximated in all cases by straight lines.

The plot of $20 \log_{10} K_o$ is a horizontal straight line because K_o is not a function of frequency. The value of this term is positive for $K_o > 1$, zero for $K_o = 1$, and negative for $K_o < 1$.

Two straight lines approximate the plot of $20 \log_{10} |1 + j\omega/z_1|$. For small values of ω, the magnitude $|1 + j\omega/z_1|$ is approximately 1, and therefore

$$20 \log_{10} |1 + j\omega/z_1| \rightarrow 0 \quad \text{as } \omega \rightarrow 0. \tag{E.11}$$

For large values of ω, the magnitude $|1 + j\omega/z_1|$ is approximately ω/z_1, and therefore

$$20 \log_{10} |1 + j\omega/z_1| \rightarrow 20 \log_{10}(\omega/z_1) \quad \text{as } \omega \rightarrow \infty. \tag{E.12}$$

On a log frequency scale, $20 \log_{10}(\omega/z_1)$ is a straight line with a slope of 20 dB/decade (a decade is a 10-to-1 change in frequency). This straight line intersects the 0 dB axis at $\omega = z_1$. This value of ω is called the **corner frequency**. Thus, on the basis of Eqs. E.11 and E.12, two straight lines can approximate the amplitude plot of a first-order zero, as shown in Fig. E.1.

The plot of $-20 \log_{10} \omega$ is a straight line having a slope of 20 dB/decade that intersects the 0 dB axis at $\omega = 1$. Two straight lines approximate the plot of $-20 \log_{10} |1 + j\omega/p_1|$. Here the two straight lines intersect on the 0 dB axis at $\omega = p_1$. For large values of ω, the straight line $20 \log_{10}(\omega/p_1)$ has a slope of -20 dB/decade. Figure E.2 shows the straight-line approximation of the amplitude plot of a first-order pole.

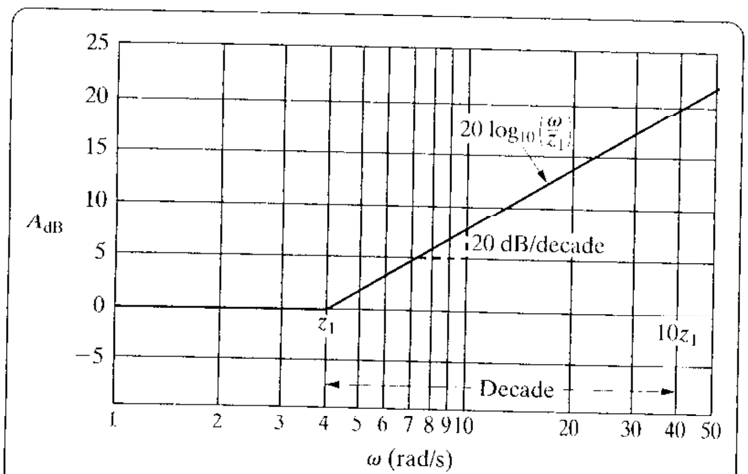

Figure E.1 A straight-line approximation of the amplitude plot of a first-order zero.

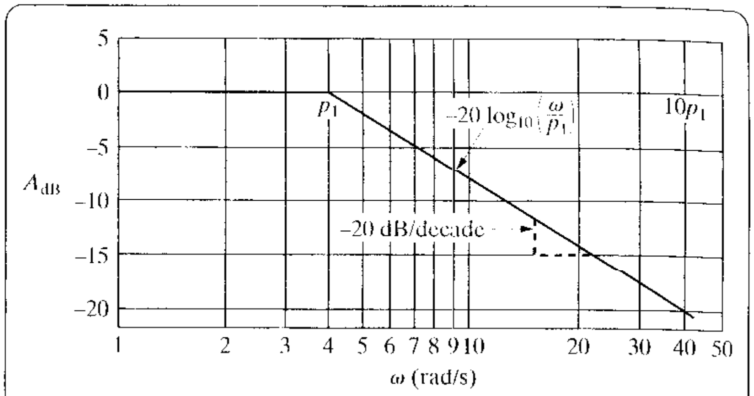

Figure E.2 A straight-line approximation of the amplitude plot of a first-order pole.

Figure E.3 shows a plot of Eq. E.10 for $K_o = \sqrt{10}$, $z_1 = 0.1$ rad/s, and $p_1 = 5$ rad/s. Each term in Eq. E.10 is labeled on Fig. E.3, so you can verify that the individual terms sum to create the resultant plot, labeled $20\log_{10}|H(j\omega)|$.

Example E.1 illustrates the construction of a straight-line amplitude plot for a transfer function characterized by first-order poles and zeros.

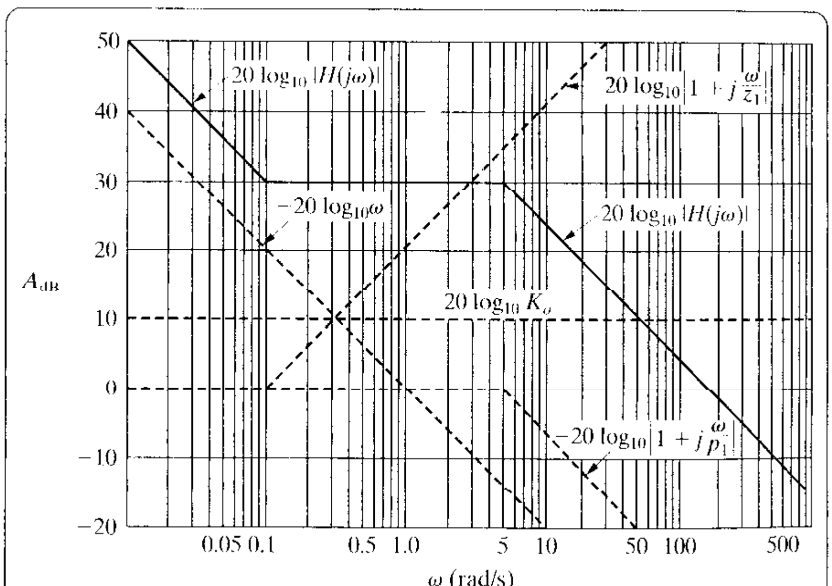

Figure E.3 A straight-line approximation of the amplitude plot for Eq. E.10.

EXAMPLE E.1

For the circuit in Fig. E.4:

a) Compute the transfer function, $H(s)$.

b) Construct a straight-line approximation of the Bode amplitude plot.

c) Calculate $20\log_{10}|H(j\omega)|$ at $\omega = 50$ rad/s and $\omega = 1000$ rad/s.

d) Plot the values computed in (c) on the straight-line graph; and

e) Suppose that $v_i(t) = 5\cos(500t + 15^\circ)$ V, and then use the Bode plot you constructed to predict the amplitude of $v_o(t)$ in the steady state.

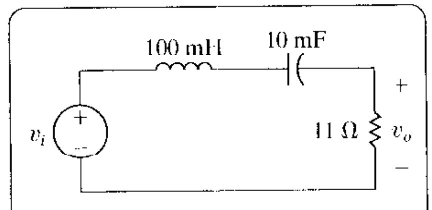

Figure E.4 The circuit for Example E.1.

SOLUTION

a) Transforming the circuit in Fig. E.4 into the s-domain and then using s-domain voltage division gives

$$H(s) = \frac{(R/L)s}{s^2 + (R/L)s + \frac{1}{LC}}.$$

Substituting the numerical values from the circuit, we get

$$H(s) = \frac{110s}{s^2 + 110s + 1000} = \frac{110s}{(s+10)(s+100)}.$$

b) We begin by writing $H(j\omega)$ in standard form:

$$H(j\omega) = \frac{0.11j\omega}{[1 + j(\omega/10)][1 + j(\omega/100)]}.$$

The expression for the amplitude of $H(j\omega)$ in decibels is

$$\begin{aligned} A_{dB} &= 20\log_{10}|H(j\omega)| \\ &= 20\log_{10}0.11 + 20\log_{10}|j\omega| \\ &\quad - 20\log_{10}\left|1 + j\frac{\omega}{10}\right| - 20\log_{10}\left|1 + j\frac{\omega}{100}\right|. \end{aligned}$$

Figure E.5 shows the straight-line plot. Each term contributing to the overall amplitude is identified.

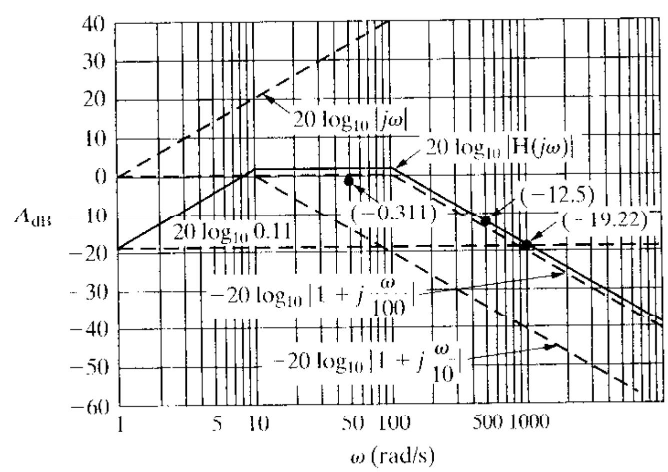

Figure E.5 The straight-line amplitude plot for the transfer function of the circuit in Fig. E.4.

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c) We have

$$H(j50) = \frac{0.11(j50)}{(1+j5)(1+j0.5)}$$
$$= 0.9648 \angle{-15.25^\circ},$$
$$20\log_{10}|H(j50)| = 20\log_{10} 0.9648$$
$$= -0.311 \text{ dB};$$
$$H(j1000) = \frac{0.11(j1000)}{(1+j100)(1+j10)}$$
$$= 0.1094 \angle{-83.72^\circ};$$
$$20\log_{10} 0.1094 = -19.22 \text{ dB}.$$

d) See Fig. E.5.

e) As we can see from the Bode plot in Fig. E.5, the value of A_{dB} at $\omega = 500$ rad/s is approximately -12.5 dB. Therefore,

$$|A| = 10^{(-12.5/20)} = 0.24$$

and

$$V_{mo} = |A|V_{mi} = (0.24)(5) = 1.19 \text{ V}.$$

We can compute the actual value of $|H(j\omega)|$ by substituting $\omega = 500$ into the equation for $|H(j\omega)|$:

$$H(j500) = \frac{0.11(j500)}{(1+j50)(1+j5)} = 0.22 \angle{-77.54^\circ}.$$

Thus, the actual output voltage magnitude for the specified signal source at a frequency of 500 rad/s is

$$V_{mo} = |A|V_{mi} = (0.22)(5) = 1.1 \text{ V}.$$

E.3 ◆ More Accurate Amplitude Plots

We can make the straight-line plots for first-order poles and zeros more accurate by correcting the amplitude values at the corner frequency, one half the corner frequency, and twice the corner frequency. At the corner frequency, the actual value in decibels is

$$A_{dB_c} = \pm 20\log_{10}|1 + j1|$$
$$= \pm 20\log_{10}\sqrt{2}$$
$$\approx \pm 3 \text{ dB}. \tag{E.13}$$

The actual value at one half the corner frequency is

$$A_{dB_{c/2}} = \pm 20\log_{10}\left|1 + j\frac{1}{2}\right|$$
$$= \pm 20\log_{10}\sqrt{5/4}$$
$$\approx \pm 1 \text{ dB}. \tag{E.14}$$

At twice the corner frequency, the actual value in decibels is

$$\begin{aligned} A_{\mathrm{dB}_{2c}} &= \pm 20 \log_{10} |1 + j2| \\ &= \pm 20 \log_{10} \sqrt{5} \\ &\approx \pm 7 \text{ dB}. \end{aligned} \quad \text{(E.15)}$$

In Eqs. E.13–E.15, the plus sign applies to a first-order zero, and the minus sign applies to a first-order pole. The straight-line approximation of the amplitude plot gives 0 dB at the corner and one half the corner frequencies, and ±6 dB at twice the corner frequency. Hence the corrections are ±3 dB at the corner frequency and ±1 dB at both one half the corner frequency and twice the corner frequency. Figure E.6 summarizes these corrections.

A 2-to-1 change in frequency is called an **octave**. A slope of 20 dB/decade is equivalent to 6.02 dB/octave, which for graphical purposes is equivalent to 6 dB/octave. Thus the corrections enumerated correspond to one octave below and one octave above the corner frequency.

If the poles and zeros of $H(s)$ are well separated, inserting these corrections into the overall amplitude plot and achieving a reasonably accurate curve is relatively easy. However, if the poles and zeros are close together, the overlapping corrections are difficult to evaluate, and you're better off using the straight-line plot as a first estimate of the amplitude characteristic. Then use a computer to refine the calculations in the frequency range of interest.

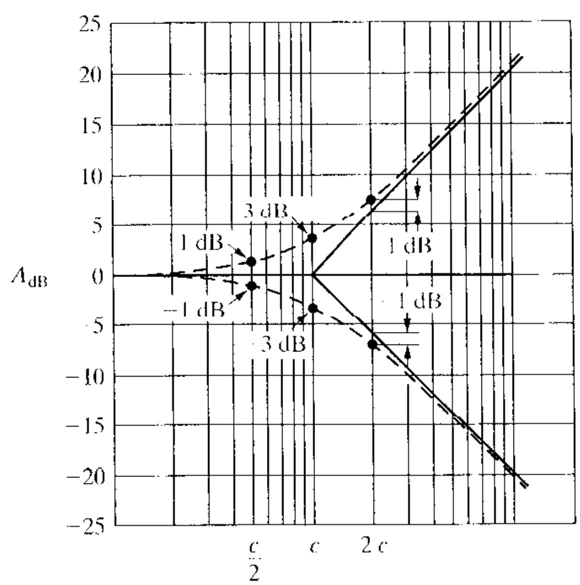

Figure E.6 Corrected amplitude plots for a first-order zero and pole.

E.4 ◆ Straight-Line Phase Angle Plots

We can also make phase angle plots by using straight-line approximations. The phase angle associated with the constant K_o is zero, and the phase angle associated with a first-order zero or pole at the origin is a constant $\pm 90°$. For a first-order zero or pole not at the origin, the straight-line approximations are as follows:

- For frequencies less than one tenth the corner frequency, the phase angle is assumed to be zero.
- For frequencies greater than 10 times the corner frequency, the phase angle is assumed to be $\pm 90°$.
- Between one tenth the corner frequency and 10 times the corner frequency, the phase angle plot is a straight line that goes through $0°$ at one-tenth the corner frequency, $\pm 45°$ at the corner frequency, and $\pm 90°$ at 10 times the corner frequency.

In all these cases, the plus sign applies to the first-order zero and the minus sign to the first-order pole. Figure E.7 depicts the straight-line approximation for a first-order zero and pole. The dashed curves show the exact variation of the phase angle as the frequency varies. Note how closely the straight-line plot approximates the actual variation in phase angle. The maximum deviation between the straight-line plot and the actual plot is approximately $6°$.

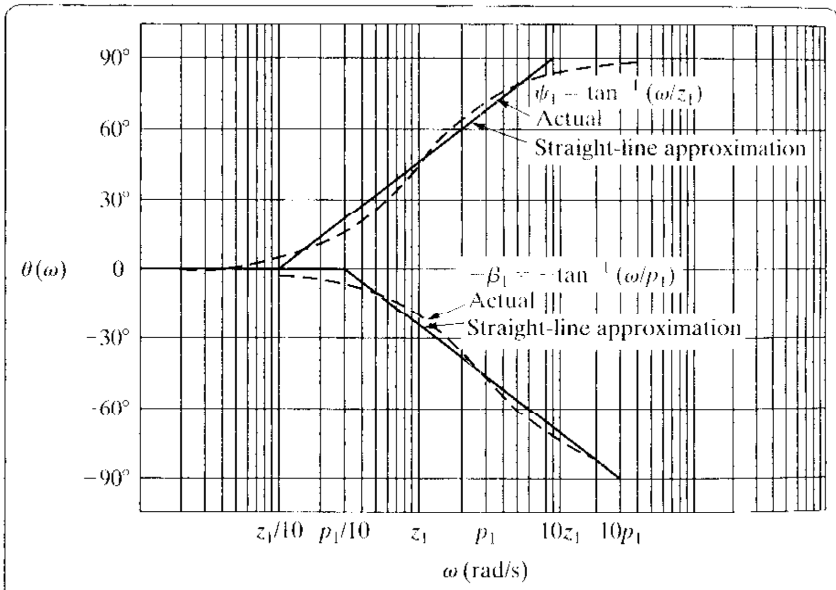

Figure E.7 Phase angle plots for a first-order zero and pole.

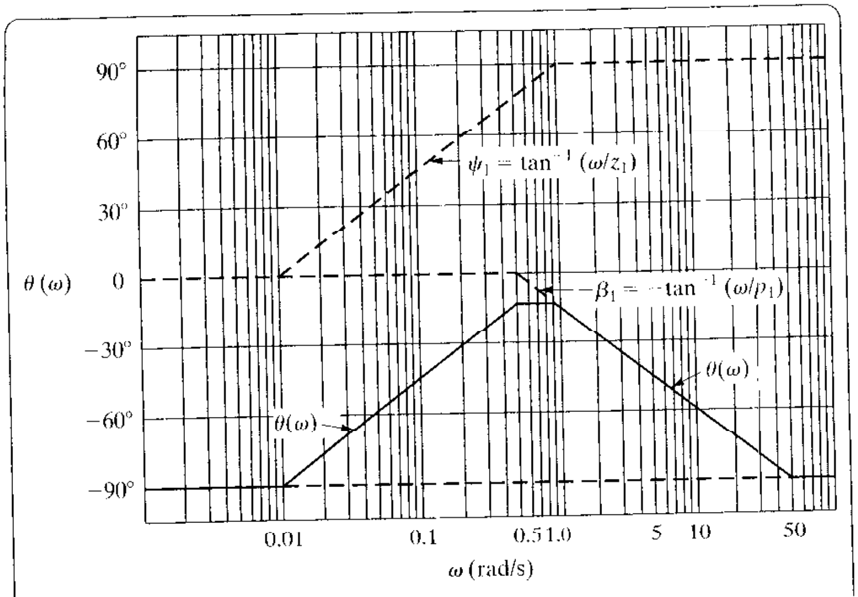

Figure E.8 A straight-line approximation of the phase angle plot for Eq. B.1.

Figure E.8 depicts the straight-line approximation of the phase angle of the transfer function given by Eq. B.1. Equation B.6 gives the equation for the phase angle; the plot corresponds to $z_1 = 0.1$ rad/s, and $p_1 = 5$ rad/s.

An illustration of a phase angle plot using a straight-line approximation is given in Example E.2.

EXAMPLE E.2

a) Make a straight-line phase angle plot for the transfer function in Example E.1.

b) Compute the phase angle $\theta(\omega)$ at $\omega = 50, 500$, and 1000 rad/s.

c) Plot the values of (b) on the diagram of (a).

d) Using the results from Example E.1(e) and (b) of this example, compute the steady-state output voltage if the source voltage is given by $v_i(t) = 10\cos(500t - 25^\circ)$ V.

SOLUTION

a) From Example E.1,

$$H(j\omega) = \frac{0.11(j\omega)}{[1 + j(\omega/10)][1 + j(\omega/100)]}$$

$$= \frac{0.11|j\omega|}{|1 + j(\omega/10)||1 + j(\omega/100)|}\angle(\psi_1 - \beta_1 - \beta_2).$$

Continued ◆

Therefore,

$$\theta(\omega) = \psi_1 - \beta_1 - \beta_2,$$

where $\psi_1 = 90^\circ$, $\beta_1 = \tan^{-1}(\omega/10)$, and $\beta_2 = \tan^{-1}(\omega/100)$. Figure E.9 depicts the straight-line approximation of $\theta(\omega)$.

b) We have

$$H(j50) = 0.96 \angle -15.25^\circ,$$

$$H(j500) = 0.22 \angle -77.54^\circ,$$

$$H(j1000) = 0.11 \angle -83.72^\circ.$$

Thus,

$$\theta(j50) = -15.25^\circ,$$

$$\theta(j500) = -77.54^\circ,$$

and

$$\theta(j1000) = -83.72^\circ.$$

c) See Fig. E.9.

d) We have

$$V_{mo} = |H(j500)|V_{mi}$$

$$= (0.22)(10)$$

$$= 2.2 \text{ V},$$

and

$$\theta_o = \theta(\omega) + \theta_i$$

$$= -77.54^\circ - 25^\circ$$

$$= -102.54^\circ.$$

Thus,

$$v_o(t) = 2.2\cos(500t - 102.54^\circ) \text{ V}.$$

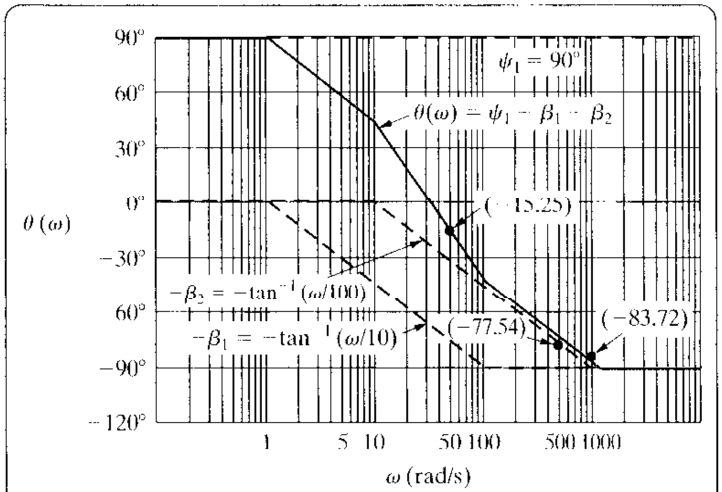

Figure E.9 A straight-line approximation of $\theta(\omega)$ for Example E.2.