REVIEW QUESTIONS

Q7.4 An elliptically polarized wave is characterized by amplitudes a_x and a_y and by the phase difference δ. If a_x and a_y are both nonzero, what should δ be in order for the polarization state to reduce to linear polarization?

Q7.5 Which of the following two descriptions defines an RHC polarized wave: A wave incident upon an observer is RHC polarized if its electric field appears to the observer to rotate in a counterclockwise direction (a) as a function of time in a fixed plane perpendicular to the direction of wave travel or (b) as a function of travel distance at a fixed time t?

EXERCISE 7.5 The electric field of a plane wave is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\,3\cos(\omega t - kz) + \hat{\mathbf{y}}\,4\cos(\omega t - kz) \qquad \text{(V/m)}.$$

Determine (a) the polarization state, (b) the modulus of $\mathbf{E}$, and (c) the inclination angle.

Ans. (a) Linear, (b) $|\mathbf{E}| = 5\cos(\omega t - kz)$ (V/m), (c) $\psi_0 = 53.1°$. (See CD)

EXERCISE 7.6 If the electric field phasor of a TEM wave is given by $\widetilde{\mathbf{E}} = (\hat{\mathbf{y}} - \hat{\mathbf{z}}j)e^{-jkx}$, determine the polarization state.

Ans. RHC polarization. (See CD)

CD D7.1-7.5

7-4 Plane-Wave Propagation in Lossy Media

To examine wave propagation in a conducting medium, we return to the wave equation given by Eq. (7.15),

$$\nabla^2\widetilde{\mathbf{E}} - \gamma^2\widetilde{\mathbf{E}} = 0, \tag{7.61}$$

with

$$\gamma^2 = -\omega^2\mu\varepsilon_c = -\omega^2\mu(\varepsilon' - j\varepsilon''), \tag{7.62}$$

where $\varepsilon' = \varepsilon$ and $\varepsilon'' = \sigma/\omega$. Since γ is complex, we express it as

$$\gamma = \alpha + j\beta, \tag{7.63}$$

where α is the ***attenuation constant*** of the medium and β is its ***phase constant***. By replacing γ with $(\alpha + j\beta)$ in Eq. (7.62), we have

$$\begin{aligned}(\alpha + j\beta)^2 &= (\alpha^2 - \beta^2) + j2\alpha\beta \\ &= -\omega^2\mu\varepsilon' + j\omega^2\mu\varepsilon''.\end{aligned} \tag{7.64}$$

The rules of complex algebra require the real and imaginary parts on one side of an equation to be respectively equal to the real and imaginary parts on the other side. Hence,

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon', \tag{7.65a}$$

$$2\alpha\beta = \omega^2\mu\varepsilon''. \tag{7.65b}$$

Solving these two equations for α and β gives

$$\alpha = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]\right\}^{1/2} \quad \text{(Np/m)}, \tag{7.66a}$$

$$\beta = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]\right\}^{1/2} \quad \text{(rad/m)}. \tag{7.66b}$$

For a uniform plane wave with an electric field $\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\,\widetilde{E}_x(z)$ traveling in the $+z$-direction, the wave equation given by Eq. (7.61) reduces to

$$\frac{d^2\widetilde{E}_x(z)}{dz^2} - \gamma^2\,\widetilde{E}_x(z) = 0. \tag{7.67}$$

The solution of this wave equation leads to

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x(z) = \hat{\mathbf{x}}E_{x0}e^{-\gamma z} = \hat{\mathbf{x}}E_{x0}e^{-\alpha z}e^{-j\beta z}. \quad (7.68)$$

The associated magnetic field $\widetilde{\mathbf{H}}$ can be determined either (1) by applying Eq. (7.2b): $\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$, (2) by applying Eq. (7.39a): $\widetilde{\mathbf{H}} = (\hat{\mathbf{k}} \times \widetilde{\mathbf{E}})/\eta_c$, where η_c is the ***intrinsic impedance of the lossy medium***, or (3) by analogy with the lossless case. Any one of these approaches gives

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\,\widetilde{H}_y(z) = \hat{\mathbf{y}}\,\frac{\widetilde{E}_x(z)}{\eta_c} = \hat{\mathbf{y}}\frac{E_{x0}}{\eta_c}e^{-\alpha z}e^{-j\beta z}, \quad (7.69)$$

where

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \quad (\Omega). \quad (7.70)$$

We noted earlier that in a nonconducting medium, $\mathbf{E}(z, t)$ is in phase with $\mathbf{H}(z, t)$, but because η_c is a complex quantity in a conducting medium, the fields no longer have equal phase (as will be illustrated in Example 7-4).

From Eq. (7.68), the magnitude of $\widetilde{E}_x(z)$ is given by

$$|\widetilde{E}_x(z)| = |E_{x0}e^{-\alpha z}e^{-j\beta z}| = |E_{x0}|e^{-\alpha z}, \quad (7.71)$$

which decreases exponentially with z at a rate specified by the attenuation constant α. Since $\widetilde{H}_y = \widetilde{E}_x/\eta_c$, the magnitude of $\widetilde{H}_y$ also attenuates as $e^{-\alpha z}$. The attenuation process converts part of the energy carried by the electromagnetic wave into heat as a result of conduction in the medium. Through a distance $z = \delta_s$ such that

$$\delta_s = \frac{1}{\alpha} \quad \text{(m)}, \quad (7.72)$$

the wave magnitude decreases by a factor of $e^{-1} \approx 0.37$ compared with its value at $z = 0$, as shown in Fig. 7-13. This distance δ_s, called the ***skin depth*** of the medium, characterizes how well an electromagnetic wave can

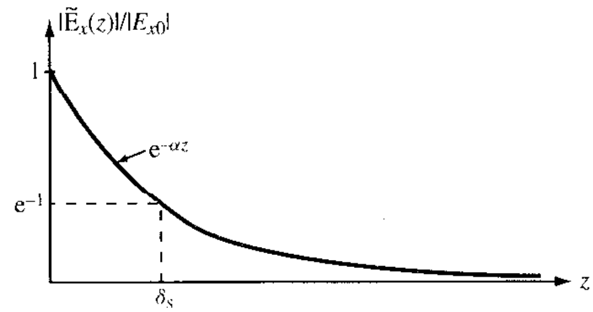

Figure 7-13: Attenuation of the magnitude of $\widetilde{E}_x(z)$ with distance z. The skin depth δ_s is the value of z at which $|\widetilde{E}_x(z)|/|E_{x0}| = e^{-1}$, or $z = \delta_s = 1/\alpha$.

D7.6-7.8

penetrate into a conducting medium. In a perfect dielectric, $\sigma = 0$; hence, $\alpha = 0$ and therefore $\delta_s = \infty$. Thus, in free space, a plane wave can propagate with no loss in magnitude indefinitely. On the other extreme, if the medium is a perfect conductor with $\sigma = \infty$, use of $\varepsilon'' = \sigma/\omega$ in Eq. (7.66a) leads to $\alpha = \infty$ and hence $\delta_s = 0$. In a coaxial cable, if the outer conductor is designed to be several skin depths thick, it serves to prevent the energy inside the cable from leaking outward, as well as to shield against the penetration of outside electromagnetic energy into the cable.

The expressions given by Eqs. (7.66a), (7.66b), and (7.70) for α, β, and η_c are valid for any linear, isotropic, and homogeneous medium. If the medium is a perfect dielectric ($\sigma = 0$), these expressions reduce to the lossless case [Section 7-2], wherein $\alpha = 0$, $\beta = k = \omega\sqrt{\mu\varepsilon}$, and $\eta_c = \eta$. For a lossy medium, the ratio $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon$ appears in all these expressions and plays an important role in determining how lossy a medium is. When $\varepsilon''/\varepsilon' \ll 1$, the medium is called a ***low-loss dielectric***, and when $\varepsilon''/\varepsilon' \gg 1$, the medium is characterized as a ***good conductor***. In practice, the medium may be regarded as a low-loss dielectric if

$\varepsilon''/\varepsilon' < 10^{-2}$, as a good conductor if $\varepsilon''/\varepsilon' > 10^2$, and as a ***quasi-conductor*** if $10^{-2} \le \varepsilon''/\varepsilon' \le 10^2$.

7-4.1 Low-Loss Dielectric

From Eq. (7.62), the general expression for γ is given by

$$\gamma = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{1/2}. \quad (7.73)$$

For any quantity $|x| \ll 1$, the function $(1-x)^{1/2}$ can be approximated by the first two terms of its binomial series; that is, $(1-x)^{1/2} \simeq 1 - x/2$. By applying such an expansion to Eq. (7.73) for a low-loss dielectric with $x = j\varepsilon''/\varepsilon'$ and $\varepsilon''/\varepsilon' \ll 1$, we have

$$\gamma \cong j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{2\varepsilon'}\right). \quad (7.74)$$

The real and imaginary parts of Eq. (7.74) give

$$\alpha \cong \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \quad \text{(Np/m)}, \quad (7.75a)$$

$$\beta \cong \omega\sqrt{\mu\varepsilon'} = \omega\sqrt{\mu\varepsilon} \quad \text{(rad/m)}. \quad (7.75b)$$

We note that the expression for β is the same as that for the wavenumber k of a lossless medium. Applying the binomial approximation $(1-x)^{-1/2} \simeq (1 + x/2)$ to Eq. (7.70) leads to

$$\eta_c \cong \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right) = \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2\omega\varepsilon}\right). \quad (7.76a)$$

In practice, these approximate expressions for α, β, and η_c are used whenever $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 1/100$, in which case the second term in Eq. (7.76a) may be ignored. Thus,

$$\eta_c \cong \sqrt{\frac{\mu}{\varepsilon}}, \quad (7.76b)$$

which is the same as the expression given by Eq. (7.31) for the lossless case.

7-4.2 Good Conductor

We now examine the case of a good conductor characterized by $\varepsilon''/\varepsilon' > 100$. Under this condition, Eqs. (7.66a), (7.66b), and (7.70) can be approximated as

$$\alpha \cong \omega\sqrt{\frac{\mu\varepsilon''}{2}} = \omega\sqrt{\frac{\mu\sigma}{2\omega}} = \sqrt{\pi f\mu\sigma} \quad \text{(Np/m)}, \quad (7.77a)$$

$$\beta = \alpha \simeq \sqrt{\pi f\mu\sigma} \quad \text{(rad/m)}, \quad (7.77b)$$

$$\eta_c \cong \sqrt{j\frac{\mu}{\varepsilon''}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma} \quad (\Omega). \quad (7.77c)$$

In Eq. (7.77c), we used the relation given by Eq. (1.53): $\sqrt{j} = (1+j)/\sqrt{2}$. For a perfect conductor with $\sigma = \infty$, these expressions give $\alpha = \beta = \infty$, and $\eta_c = 0$. A perfect conductor is equivalent to a short circuit.

Expressions for the propagation parameters in various types of media are summarized in Table 7-1 for easy reference.

Example 7-4 Plane Wave in Seawater

A uniform plane wave is traveling downward in the $+z$-direction in seawater, with the x–y plane denoting the sea surface and $z = 0$ denoting a point just below the surface. The constitutive parameters of seawater are $\varepsilon_r = 80$, $\mu_r = 1$, and $\sigma = 4$ S/m. If the magnetic field at $z = 0$ is given by $\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 100\cos(2\pi \times 10^3 t + 15^\circ)$ (mA/m),

(a) obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$, and

(b) determine the depth at which the amplitude of $\mathbf{E}$ is 1% of its value at $z = 0$.

Solution: (a) Since $\mathbf{H}$ is along $\hat{\mathbf{y}}$ and the propagation direction is $\hat{\mathbf{z}}$, $\mathbf{E}$ must be along $\hat{\mathbf{x}}$. Hence, the general expressions for the phasor fields are

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_{x0} e^{-\alpha z} e^{-j\beta z}, \quad (7.78a)$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z} \quad (7.78b)$$

Table 7-1: Expressions for α, β, η_c, u_p, and λ for various types of media.

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right]\right]^{1/2}$	0	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu\sigma}$	(Np/m)
$\beta =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}+1\right]\right]^{1/2}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu\sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$(1+j)\frac{\alpha}{\sigma}$	(Ω)
$u_p =$	ω/β	$1/\sqrt{\mu\varepsilon}$	$1/\sqrt{\mu\varepsilon}$	$\sqrt{4\pi f/\mu\sigma}$	(m/s)
$\lambda =$	$2\pi/\beta = u_p/f$	u_p/f	u_p/f	u_p/f	(m)

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$.

To determine α, β, and η_c for seawater, we begin by evaluating the ratio $\varepsilon''/\varepsilon'$. From the argument of the cosine function of $\mathbf{H}(0, t)$, we deduce that $\omega = 2\pi \times 10^3$ (rad/s), and therefore $f = 1$ kHz. Hence,

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{4}{2\pi \times 10^3 \times 80 \times (10^{-9}/36\pi)} = 9 \times 10^5.$$

Since $\varepsilon''/\varepsilon' \gg 1$, seawater is a good conductor at 1 kHz. This allows us to use the good-conductor expressions given in Table 7-1:

$$\alpha = \sqrt{\pi f \mu\sigma} = \sqrt{\pi \times 10^3 \times 4\pi \times 10^{-7} \times 4} = 0.126 \quad \text{(Np/m)}, \qquad (7.79a)$$

$$\beta = \alpha = 0.126 \quad \text{(rad/m)}, \qquad (7.79b)$$

$$\eta_c = (1+j)\frac{\alpha}{\sigma} = (\sqrt{2}\, e^{j\pi/4})\frac{0.126}{4} = 0.044 e^{j\pi/4} \quad (\Omega). \qquad (7.79c)$$

As no explicit information has been given about the electric field amplitude E_{x0}, we should assume it to be complex; that is, $E_{x0} = |E_{x0}|e^{j\phi_0}$. The wave's instantaneous electric and magnetic fields are then given by

$$\mathbf{E}(z,t) = \mathfrak{Re}\left[\hat{\mathbf{x}}|E_{x0}|e^{j\phi_0}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right] = \hat{\mathbf{x}}|E_{x0}|e^{-0.126z}\cos(2\pi \times 10^3 t - 0.126z + \phi_0) \quad \text{(V/m)}, \qquad (7.80a)$$

$$\mathbf{H}(z,t) = \mathfrak{Re}\left[\hat{\mathbf{y}}\frac{|E_{x0}|e^{j\phi_0}}{0.044e^{j\pi/4}}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right] = \hat{\mathbf{y}}22.5|E_{x0}|e^{-0.126z}\cos(2\pi \times 10^3 t - 0.126z + \phi_0 - 45^\circ) \quad \text{(A/m)}. \qquad (7.80b)$$

At $z = 0$,

$$\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 22.5|E_{x0}| \cos(2\pi \times 10^3 t + \phi_0 - 45°) \quad \text{(A/m)}. \tag{7.81}$$

By comparing Eq. (7.81) with the expression given in the problem statement,

$$\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 100 \cos(2\pi \times 10^3 t + 15°) \qquad \text{(mA/m)},$$

we deduce that

$$22.5|E_{x0}| = 100 \times 10^{-3}$$

or

$$|E_{x0}| = 4.44 \qquad \text{(mV/m)},$$

and

$$\phi_0 - 45° = 15° \qquad \text{or} \qquad \phi_0 = 60°.$$

Hence, the final expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ are

$$\mathbf{E}(z, t) = \hat{\mathbf{x}}\, 4.44 e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 60°) \quad \text{(mV/m)}, \tag{7.82a}$$

$$\mathbf{H}(z, t) = \hat{\mathbf{y}}\, 100 e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 15°) \quad \text{(mA/m)}. \tag{7.82b}$$

(b) The depth at which the amplitude of **E** has decreased to 1% of its initial value at $z = 0$ is obtained from

$$0.01 = e^{-0.126z} \qquad \text{or} \qquad z = \frac{\ln(0.01)}{-0.126} = 36 \text{ m}. \quad \blacksquare$$

EXERCISE 7.7 The constitutive parameters of copper are $\mu = \mu_0 = 4\pi \times 10^{-7}$ (H/m), $\varepsilon = \varepsilon_0 \simeq (1/36\pi) \times 10^{-9}$ (F/m), and $\sigma = 5.8 \times 10^7$ (S/m). Assuming that these parameters are frequency independent, over what frequency range of the electromagnetic spectrum [see Fig. 1-15] is copper a good conductor?

Ans. $f < 1.04 \times 10^{16}$ Hz, which includes the radio spectrum, the infrared and visible regions, and part of the ultraviolet region. (See ❀)

EXERCISE 7.8 Over what frequency range may dry soil, with $\varepsilon_r = 3$, $\mu_r = 1$, and $\sigma = 10^{-4}$ (S/m), be regarded as a low-loss dielectric medium?

Ans. $f > 60$ MHz. (See ❀)

EXERCISE 7.9 For a wave traveling in a medium with a skin depth δ_s, what is the amplitude of **E** at a distance of $3\delta_s$ compared with its initial value?

Ans. $e^{-3} \approx 0.05$ or 5%. (See ❀)

7-5 Current Flow in a Good Conductor

When a d-c voltage is connected across the ends of a conducting wire, the current flowing through the wire has a uniform current density **J** over the wire's cross section. That is, **J** has the same value along the axis of the wire as along its outer perimeter [Fig. 7-14(a)]. This is not true in the a-c case. As we will see shortly, the current density

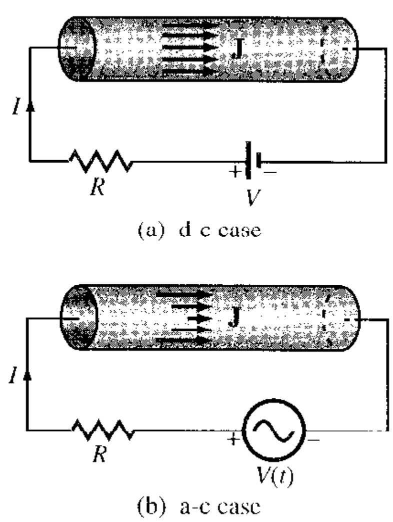

Figure 7-14: Current density **J** in a conducting wire is (a) uniform across its cross section in the d-c case, but (b) in the a-c case, **J** is highest along the wire's perimeter.