ECE 3300 Maxwell's Equations in Free Space and Plane Waves

Maxwell's Equations (General – Time Domain and Phasor Forms)

(To convert from time domain to frequency/phasor domain, assume that E,H are functions of $e^{j\omega t}$. Then d/dt = jω)

$$\nabla \bullet \overline{D} = \rho_v$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = -j\omega\mu\overline{H}$$

$$\nabla \bullet \overline{B} = 0$$

$$\nabla \times \overline{H} = \overline{J_c} + \frac{\partial \overline{D}}{\partial t} = \sigma\overline{E} + j\omega\varepsilon\overline{E} = j\omega\varepsilon_c\overline{E}$$

$$\varepsilon_c = \varepsilon_o\varepsilon_r - j\sigma/\omega = \varepsilon' - j\varepsilon''$$

Here, Jc = conduction current density (A/m^2), ε_c is complex dielectric constant.

Wave Equations in Charge-Free / Current-free Medium:
If the material is lossless (σ=0), then there is no charge, and no current. This is called a "perfect dielectric". An insulator is a good or near-perfect dielectric.

$$\nabla \bullet \overline{E} = 0$$

$$\nabla \times \overline{E} = -j\omega\mu\overline{H}$$

$$\nabla \bullet \overline{H} = 0$$

$$\nabla \times \overline{H} = j\omega\varepsilon_c\overline{E}$$

Your book derives the general wave equation in the time domain. So we will derive the wave equation in the frequency (phasor) domain instead ….

$$Derivation \quad of \quad WaveEquation:$$

$$Take \quad \nabla\times:$$

$$\nabla\times(\nabla\times\overline{E}) = -j\omega\mu(\nabla\times\overline{H})$$

$$Substitute:$$

$$\nabla\times(\nabla\times\overline{E}) = -j\omega\mu(j\omega\varepsilon_c\overline{E}) = \omega^2\mu\varepsilon_c\overline{E}$$

$$Vector\_property:$$

$$\nabla\times(\nabla\times\overline{E}) = \nabla(\nabla\bullet\overline{E}) - \nabla^2\overline{E}$$

$$Substitute:$$

$$\nabla\times(\nabla\times\overline{E}) = \nabla(0) - \nabla^2\overline{E} = -\nabla^2\overline{E}$$

$$Combine:$$

$$\omega^2\mu\varepsilon_c\overline{E} - \nabla^2\overline{E} = 0$$

$$\nabla^2\overline{E} - \gamma^2\overline{E} = 0$$

$$\gamma^2 = \omega^2\mu\varepsilon_c = \alpha + j\beta$$

$$Lossless: k^2 = \omega^2\mu\varepsilon = \beta^2$$

$$\nabla^2\overline{E} - k^2\overline{E} = 0$$

$$Similarly:$$

$$\nabla^2\overline{H} - \gamma^2\overline{H} = 0$$

These wave equations can be used to describe any type of wave (planar, spherical, etc.). No approximations or limitations have been made (yet).

Plane Waves are a special type of wave:

- Planar wavefront of infinite extent
- TEM: E,H are perpendicular to eachother and to the direction of propagation
- Plane waves are like transmission line waves (which are also TEM). Look for the similarities.

Here are the transmission line wave equations:

$-d^2 I(z)/dz^2 - \gamma^2 I(z) = 0$

$-d^2 V(z)/dz^2 - \gamma^2 V(z) = 0$

How do we solve problems with the wave equation:

1) Write wave equation (for instance in rectangular coordinates).
2) Separate the vector components into x,y,z. "Equate vector components", which means to do a separate equation for each vector component.
3) For plane wave front (flat wavefront) moving in the z-direction, d/dx and d/dy = 0. Substitute this into (2) above.
4) Verify that Ez or Hz = 0. PLANE WAVE HAS NO FIELD COMPONENTS in direction of propagation. It is a TEM wave like the ones we have studied in transmission lines.

General solution for a TEM wave: (such as Plane wave)
This applies to vector fields AND to each component separately.

$$\overline{E}(z) = \overline{E}^+(z) + \overline{E}^-(z) = \overline{E_0}^+ e^{-j\gamma z} + \overline{E}_0^-(z) e^{j\gamma z}$$

Now, for a plane-wave traveling in the z-direction, Ez=0, but Ex and Ey may not be zero.
Solve for **H** from Maxwell's equations (see text), and we obtain:

$$H_{y0}^+ = \frac{\gamma}{\omega\mu} E_{xo}^+ = \frac{1}{\eta} E_{xo}^+$$

Where η is **Intrinsic Impedance** of medium.
η=377 for air
η=real for lossless materials (E and H are perpendicular and in-phase)
η=complex for lossy materials. (E and H are perpendicular an out-of-phase)

Duality to Transmission Lines:

E → V (recall also V = - integral E dl)
H → I (recall also I = closed integral H dl)
η → Zo

Other Similarities:

1) Waves can be broken into + and – traveling waves
2) Propagate in z-direction as e^{-jkz} (k=β), including effect of phase change and attentuation
3) Waves will reflect, depending on the material (discontinuity) they hit.
   (a) Reflection coefficient, transmission coefficient
   (b) Standing Waves
   (c) Smith Chart solution

Differences:

1) V and I are not vector quantities, E and H are. This means we can have "polarization".