and the instantaneous voltage $v(z, t)$ is

$$
\begin{aligned}
v(z,t) &= \Re\mathfrak{e}[\widetilde{V}(z)e^{j\omega t}] \\
&= 10.2\cos(\omega t - \beta z + 159^\circ) \\
&\quad + 4.55\cos(\omega t + \beta z + 185.6^\circ) \qquad \text{(V)}.
\end{aligned}
$$

Similarly, use of V_0^+ in Eq. (2.51b) leads to

$$
\begin{aligned}
\widetilde{I}(z) &= 0.20e^{j159^\circ}(e^{-j\beta z} - 0.45e^{j26.6^\circ}e^{j\beta z}), \\
i(z,t) &= 0.20\cos(\omega t - \beta z + 159^\circ) \\
&\quad + 0.091\cos(\omega t + \beta z + 5.6^\circ) \qquad \text{(A)}. \quad \blacksquare
\end{aligned}
$$

M2.4

2-7 Special Cases of the Lossless Line

We often encounter situations involving lossless transmission lines with particular terminations or lines whose lengths exhibit particularly useful properties. We shall now consider some of these special cases.

2-7.1 Short-Circuited Line M2.1D

The transmission line shown in Fig. 2-15(a) is terminated in a short circuit, $Z_L = 0$. Consequently, the voltage reflection coefficient defined by Eq. (2.49a) is $\Gamma = -1$, and the voltage standing-wave ratio given by Eq. (2.59) is $S = \infty$. From Eqs. (2.51a) and (2.51b), the voltage and current on a short-circuited lossless transmission line are given by

$$\widetilde{V}_{sc}(z) = V_0^+[e^{-j\beta z} - e^{j\beta z}] = -2jV_0^+\sin\beta z, \quad (2.67a)$$

$$\widetilde{I}_{sc}(z) = \frac{V_0^+}{Z_0}[e^{-j\beta z} + e^{j\beta z}] = \frac{2V_0^+}{Z_0}\cos\beta z. \quad (2.67b)$$

The voltage $\widetilde{V}_{sc}(z)$ is zero at the load ($z = 0$), as it should be for a short circuit, and its amplitude varies as $\sin\beta z$, whereas the current $\widetilde{I}_{sc}(z)$ is a maximum at the load and it varies as $\cos\beta z$. Both quantities are displayed in Fig. 2-15 as a function of negative z.

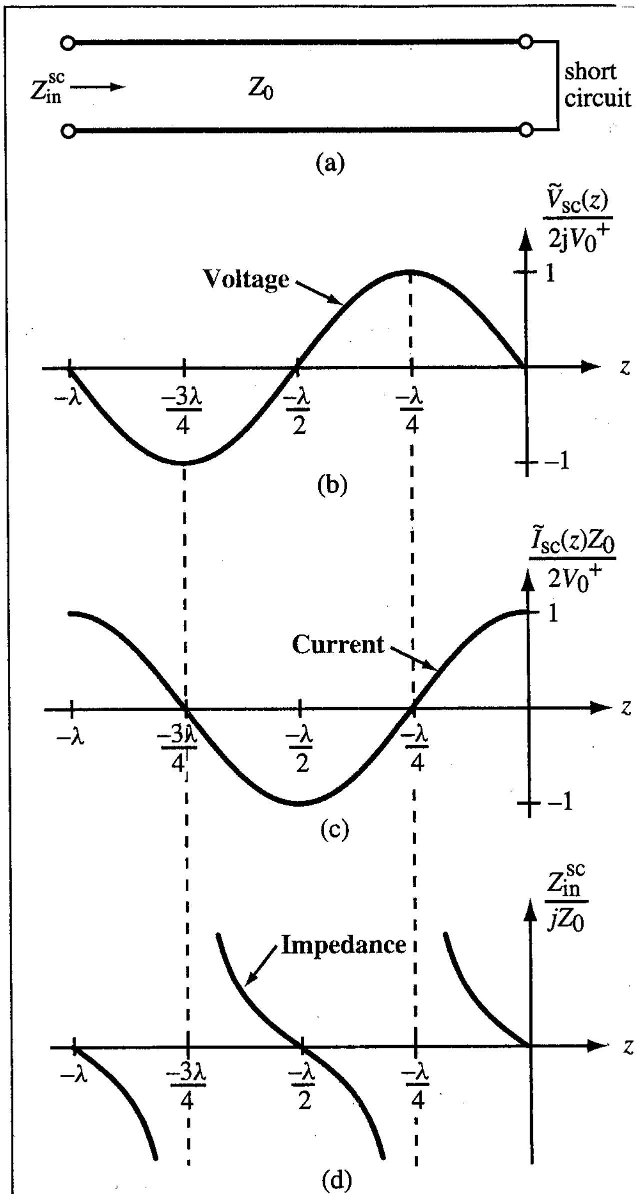

Figure 2-15: Transmission line terminated in a short circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

The input impedance of the line at $z = -l$ is given by the ratio of $\widetilde{V}_{sc}(-l)$ to $\widetilde{I}_{sc}(-l)$. Denoting Z_{in}^{sc} as the input impedance for a short-circuited line, we have

$$Z_{in}^{sc} = \frac{\widetilde{V}_{sc}(-l)}{\widetilde{I}_{sc}(-l)} = jZ_0 \tan \beta l. \quad (2.68)$$

A plot of Z_{in}^{sc}/jZ_0 versus negative z is shown in Fig. 2-15(d).

In general, the input impedance Z_{in} may consist of a real part, or input resistance R_{in}, and an imaginary part, or input reactance X_{in}:

$$Z_{in} = R_{in} + jX_{in}. \quad (2.69)$$

In the case of the short-circuited lossless line, the input impedance is purely reactive ($R_{in} = 0$). If $\tan \beta l \geq 0$, the line appears inductive, acting like an equivalent inductor L_{eq} whose impedance is equal to Z_{in}^{sc}. Thus,

$$j\omega L_{eq} = jZ_0 \tan \beta l, \qquad \text{if } \tan \beta l \geq 0, \quad (2.70a)$$

or

$$L_{eq} = \frac{Z_0 \tan \beta l}{\omega} \quad \text{(H)}. \quad (2.70b)$$

The minimum line length l that would result in an input impedance Z_{in}^{sc} equivalent to that of an inductor of inductance L_{eq} is

$$l = \frac{1}{\beta} \tan^{-1}\left(\frac{\omega L_{eq}}{Z_0}\right) \quad \text{(m)}. \quad (2.70c)$$

Similarly, if $\tan \beta l \leq 0$, the input impedance is capacitive, in which case the line acts like an equivalent capacitor C_{eq} such that

$$\frac{1}{j\omega C_{eq}} = jZ_0 \tan \beta l, \qquad \text{if } \tan \beta l \leq 0, \quad (2.71a)$$

or

$$C_{eq} = -\frac{1}{Z_0 \omega \tan \beta l} \quad \text{(F)}. \quad (2.71b)$$

Since l is a positive number, the shortest length l for which $\tan \beta l \leq 0$ corresponds to the range $\pi/2 \leq \beta l \leq \pi$. Hence, the minimum line length l that would result in an input impedance Z_{in}^{sc} equivalent to that of a capacitor of capacitance C_{eq} is

$$l = \frac{1}{\beta}\left[\pi - \tan^{-1}\left(\frac{1}{\omega C_{eq} Z_0}\right)\right] \quad \text{(m)}. \quad (2.71c)$$

These results mean that, through proper choice of the length of a short-circuited line, we can make substitutes for capacitors and inductors with any desired reactance. Such a practice is indeed common in the design of microwave circuits and high-speed integrated circuits, because making an actual capacitor or inductor is often more difficult than making a shorted transmission line.

Example 2-7 **Equivalent Reactive Elements**

Choose the length of a shorted 50-Ω lossless transmission line (Fig. 2-16) such that its input impedance at 2.25 GHz is equivalent to the reactance of a capacitor with capacitance $C_{eq} = 4$ pF. The wave velocity on the line is $0.75c$.

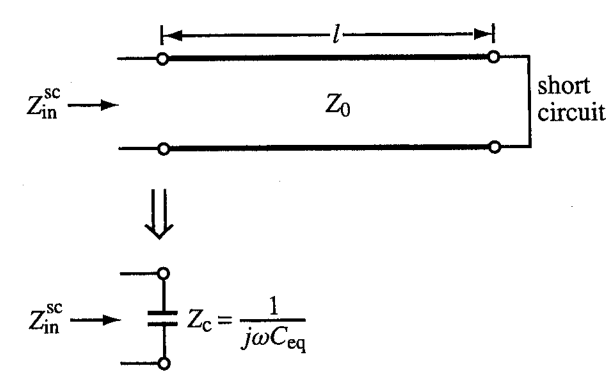

Figure 2-16: Shorted line as equivalent capacitor (Example 2-7).

Solution: We are given

$$u_p = 0.75c = 0.75 \times 3 \times 10^8 = 2.25 \times 10^8 \text{ m/s},$$
$$Z_0 = 50\ \Omega,$$
$$f = 2.25 \text{ GHz} = 2.25 \times 10^9 \text{ Hz},$$
$$C_{eq} = 4 \text{ pF} = 4 \times 10^{-12} \text{ F}.$$

The phase constant is

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{u_p} = \frac{2\pi \times 2.25 \times 10^9}{2.25 \times 10^8} = 62.8 \quad \text{(rad/m)}.$$

From Eq. (2.71a),

$$\tan \beta l = -\frac{1}{Z_0 \omega C_{eq}} = -\frac{1}{50 \times 2\pi \times 2.25 \times 10^9 \times 4 \times 10^{-12}} = -0.354.$$

The tangent function is negative when its argument is in the second or fourth quadrants. The solution for the second quadrant is

$$\beta l_1 = 2.8 \text{ rad} \quad \text{or} \quad l_1 = \frac{2.8}{\beta} = \frac{2.8}{62.8} = 4.46 \text{ cm},$$

and the solution for the fourth quadrant is

$$\beta l_2 = 5.94 \text{ rad} \quad \text{or} \quad l_2 = \frac{5.94}{62.8} = 9.46 \text{ cm}.$$

We also could have obtained the value of l_1 by applying Eq. (2.71c). The length l_2 is greater than l_1 by exactly $\lambda/2$. In fact, any length $l = 4.46 \text{ cm} + n\lambda/2$, where n is a positive integer, is also a solution. ■

2-7.2 Open-Circuited Line M2.1E

With $Z_L = \infty$, as illustrated in Fig. 2-17(a), we have $\Gamma = 1$, $S = \infty$, and the voltage, current, and input impedance are given by

$$\widetilde{V}_{oc}(z) = V_0^+[e^{-j\beta z} + e^{j\beta z}] = 2V_0^+ \cos \beta z, \qquad (2.72a)$$

$$\widetilde{I}_{oc}(z) = \frac{V_0^+}{Z_0}[e^{-j\beta z} - e^{j\beta z}] = \frac{-2jV_0^+}{Z_0} \sin \beta z, \qquad (2.72b)$$

$$Z_{in}^{oc} = \frac{\widetilde{V}_{oc}(-l)}{\widetilde{I}_{oc}(-l)} = -jZ_0 \cot \beta l. \qquad (2.73)$$

Plots of these quantities are displayed in Fig. 2-17 as a function of negative z.

2-7.3 Application of Short-Circuit and Open-Circuit Measurements

A network analyzer is a radio-frequency (RF) instrument capable of measuring the impedance of any load connected to its input terminal. When used to measure Z_{in}^{sc}, the input impedance of a lossless line terminated in a short circuit, and again Z_{in}^{oc}, the input impedance of the line when terminated in an open circuit, the combination of the two measurements can be used to determine the characteristic impedance of the line Z_0 and its phase constant β. The product of Eqs. (2.68) and (2.73) gives the result

$$Z_0 = +\sqrt{Z_{in}^{sc}\, Z_{in}^{oc}}, \qquad (2.74)$$

and the ratio of the same equations leads to

$$\tan \beta l = \sqrt{\frac{-Z_{in}^{sc}}{Z_{in}^{oc}}}. \qquad (2.75)$$

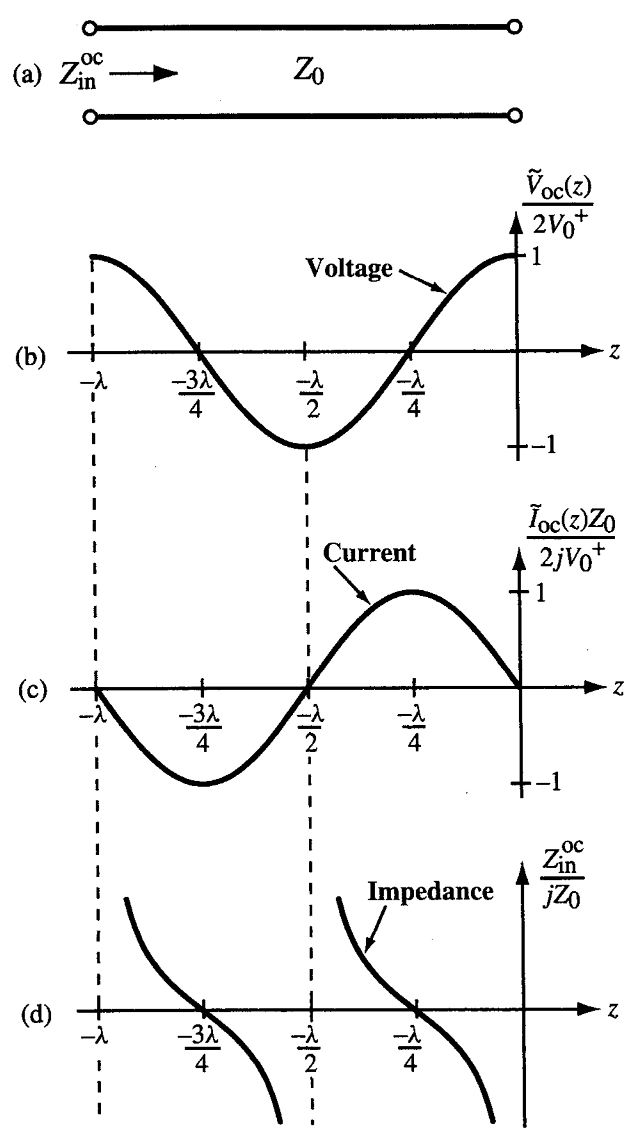

Figure 2-17: Transmission line terminated in an open circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

Because of the π phase ambiguity associated with the tangent function, the length l should be less than or equal to $\lambda/2$ to provide an unambiguous result.

Example 2-8 **Measuring Z_0 and β**

Find Z_0 and β of a 57-cm-long lossless transmission line whose input impedance was measured as $Z_{in}^{sc} = j40.42$ Ω when terminated in a short circuit and as $Z_{in}^{oc} = -j121.24$ Ω when terminated in an open circuit. From other measurements, we know that the line is between 3 and 3.25 wavelengths long.

Solution: From Eqs. (2.74) and (2.75),

$$Z_0 = \sqrt[+]{Z_{in}^{sc}\, Z_{in}^{oc}} = \sqrt{(j40.42)(-j121.24)} = 70\ \Omega,$$

$$\tan \beta l = \sqrt{\frac{-Z_{in}^{sc}}{Z_{in}^{oc}}} = \sqrt{\frac{1}{3}}\,.$$

Since l is between 3λ and 3.25λ, $\beta l = (2\pi l/\lambda)$ is between 6π radians and $(13\pi/2)$ radians. This places βl in the first quadrant (0 to $\pi/2$) in a polar coordinate system. Hence, the only acceptable solution for the above equation is $\beta l = \pi/6$ radians. This value, however, does not include the 2π multiples associated with the integer λ multiples of l. Hence, the true value of βl is

$$\beta l = 6\pi + \frac{\pi}{6} = 19.4 \qquad \text{(rad)},$$

in which case

$$\beta = \frac{19.4}{0.57} = 34 \qquad \text{(rad/m)}. \ \blacksquare$$

2-7.4 Lines of Length $l = n\lambda/2$

If $l = n\lambda/2$, where n is an integer,

$$\tan \beta l = \tan [(2\pi/\lambda)(n\lambda/2)] = \tan n\pi = 0.$$

Consequently, Eq. (2.63) reduces to

$$Z_{in} = Z_L, \qquad \text{for } l = n\lambda/2, \qquad (2.76)$$

which means that a half-wavelength line (or any integer multiple of $\lambda/2$) does not modify the load impedance. Thus, *a generator connected to a load through a half-wavelength lossless line would induce the same voltage across the load and current through it as when the line is not there.*

2-7.5 Quarter-Wave Transformer

Another case of interest is when the length of the line is a quarter-wavelength (or $\lambda/4 + n\lambda/2$, where $n = 0$ or a positive integer), corresponding to $\beta l = (2\pi/\lambda)(\lambda/4) = \pi/2$. From Eq. (2.63), the input impedance becomes

$$Z_{in} = \frac{Z_0^2}{Z_L}, \qquad \text{for } l = \lambda/4 + n\lambda/2. \qquad (2.77)$$

The utility of such a quarter-wave transformer is illustrated by the problem in Example 2-9.

Example 2-9 Quarter-Wave Transformer

A 50-Ω lossless transmission line is to be matched to a resistive load impedance with $Z_L = 100\ \Omega$ via a quarter-wave section as shown in Fig. 2-18, thereby eliminating reflections along the feedline. Find the characteristic impedance of the quarter-wave transformer.

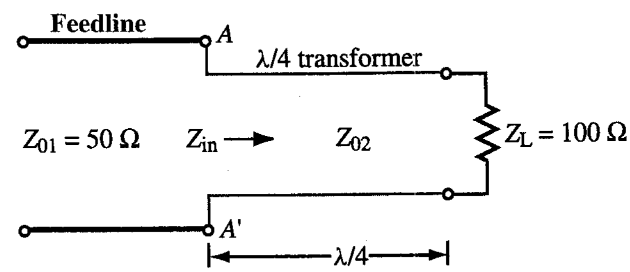

Figure 2-18: Configuration for Example 2-9.

Solution: To eliminate reflections at terminal AA', the input impedance Z_{in} looking into the quarter-wave line should be equal to Z_{01}, the characteristic impedance of the feedline. Thus, $Z_{in} = 50\ \Omega$. From Eq. (2.77),

$$Z_{in} = \frac{Z_{02}^2}{Z_L},$$

or

$$Z_{02} = \sqrt{Z_{in} Z_L} = \sqrt{50 \times 100} = 70.7\ \Omega.$$

Whereas this eliminates reflections on the feedline, it does not eliminate them on the $\lambda/4$ line. However, since the lines are lossless, all the incident power will end up getting transferred into the load Z_L. ■

2-7.6 Matched Transmission Line: $Z_L = Z_0$

For a matched lossless transmission line with $Z_L = Z_0$, (1) the input impedance $Z_{in} = Z_0$ for all locations z on the line, (2) $\Gamma = 0$, and (3) all the incident power is delivered to the load, regardless of the line length l. A summary of the properties of standing waves is given in Table 2-3.

REVIEW QUESTIONS

Q2.10 What is the difference between the characteristic impedance Z_0 and the input impedance Z_{in}? When are they the same?

Table 2-3: Properties of standing waves on a lossless transmission line.

Voltage maximum Voltage minimum	$\lvert\widetilde{V}\rvert_{\max} = \lvert V_0^+\rvert[1+\lvert\Gamma\rvert]$ $\lvert\widetilde{V}\rvert_{\min} = \lvert V_0^+\rvert[1-\lvert\Gamma\rvert]$
Positions of voltage maxima (also positions of current minima) Position of first maximum (also position of first current minimum)	$l_{\max} = \dfrac{\theta_r\lambda}{4\pi} + \dfrac{n\lambda}{2}, \quad n = 0, 1, 2, \ldots$ $l_{\max} = \begin{cases} \dfrac{\theta_r\lambda}{4\pi}, & \text{if } 0 \le \theta_r \le \pi \\ \dfrac{\theta_r\lambda}{4\pi} + \dfrac{\lambda}{2}, & \text{if } -\pi \le \theta_r \le 0 \end{cases}$
Positions of voltage minima (also positions of first current maxima) Position of first minimum (also position of first current maximum)	$l_{\min} = \dfrac{\theta_r\lambda}{4\pi} + \dfrac{(2n+1)\lambda}{4}, \quad n = 0, 1, 2, \ldots$ $l_{\min} = \dfrac{\lambda}{4}\left(1 + \dfrac{\theta_r}{\pi}\right)$
Input impedance	$Z_{\text{in}} = Z_0\left(\dfrac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l}\right)$
Positions at which Z_{in} is real	at voltage maxima and minima
Z_{in} at voltage maxima	$Z_{\text{in}} = Z_0\left(\dfrac{1+\lvert\Gamma\rvert}{1-\lvert\Gamma\rvert}\right)$
Z_{in} at voltage minima	$Z_{\text{in}} = Z_0\left(\dfrac{1-\lvert\Gamma\rvert}{1+\lvert\Gamma\rvert}\right)$
Z_{in} of short-circuited line	$Z_{\text{in}}^{\text{sc}} = jZ_0\tan\beta l$
Z_{in} of open-circuited line	$Z_{\text{in}}^{\text{oc}} = -jZ_0\cot\beta l$
Z_{in} of line of length $l = n\lambda/2$	$Z_{\text{in}} = Z_L, \quad n = 0, 1, 2, \ldots$
Z_{in} of line of length $l = \lambda/4 + n\lambda/2$ Z_{in} of matched line	$Z_{\text{in}} = Z_0^2/Z_L, \quad n = 0, 1, 2, \ldots$ $Z_{\text{in}} = Z_0$

$\lvert V_0^+\rvert$ = amplitude of incident wave, $\Gamma = \lvert\Gamma\rvert e^{j\theta_r}$ with $-\pi < \theta_r < \pi$; θ_r in radians.

D2.1-2.4