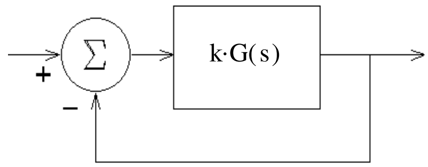
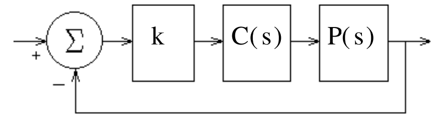
A. Stolp 2/16/06, rev 2/15/07

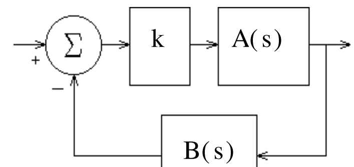
ECE 3510 Root-Locus Plots

$G(s) = \dfrac{N_G}{D_G}$ = the Open-Loop (O-L) transfer function





$G(s) = C(s)\cdot P(s)$



$G(s) = A(s)\cdot B(s)$

The Rules $(k > 1)$

1. Root-locus plots are symmetric about the real axis.
2. On the real axis, spaces left of an odd number of O-L poles and zeros are always part of the locus.
   (Essentially, every other space on the real axis (counting leftward) is part of the plot.)
3. Each O-L pole originates $(k = 0)$ one branch. (n)

   Each O-L zero terminates $(k = \infty)$ one branch. (m)

   All remaining branches go to ∞. $(n - m)$

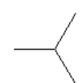
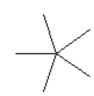
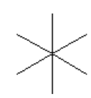
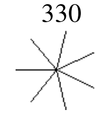
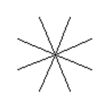
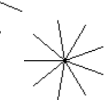
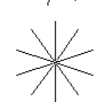
   These remaining branches approach asymptotes as they go to ∞.
4. The origin of the asymptotes is the *centroid*. centroid = $\sigma = \dfrac{\sum_{all} OLpoles - \sum_{all} OLzeros}{n - m}$

   (# poles - # zeros)
5. The angles of the asymptotes are:

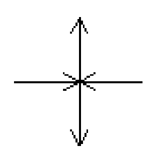
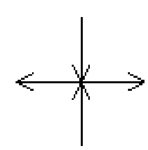
   $i\cdot\dfrac{180}{n - m}$

   where $i = 1, 3, 5, 7, 9, \ldots$ full circle

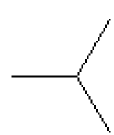
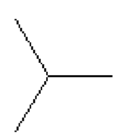
   Or figure for half circle and mirror around the real axis.

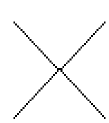
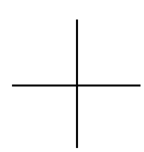
n - m	angles (degrees)						
2	90	270					
3	60	180	300				
4	45	135	225	315			
5	36	108	180	252	324		
6	30	90	150	210	270	330	
7	$\frac{180}{7}$	$3\cdot\frac{180}{7}$	$5\cdot\frac{180}{7}$	$7\cdot\frac{180}{7}$	. . .		
8	22.5	67.5	112.5	157.5	. . .		
9	20	60	100	140	180	. . .	
10	18	54	90	126	162	. . .	

6. The angles of departure (and arrival) of the locus are almost always:

    OR: 

   Only multiple poles result in different departure angles:
   (or zeros)

   triple poles:  OR:  Check real-axis rule, above

   Quadruple poles:  OR:  Check real-axis rule, above

   Multiple zeros attract branches from these same angles

ECE 3510 Root-Locus Plots **Additional Rules**

7. Breakaway points from the real axis (σ_b) are the solutions to: $\frac{d}{ds}G(s) = 0$
(and arrival)

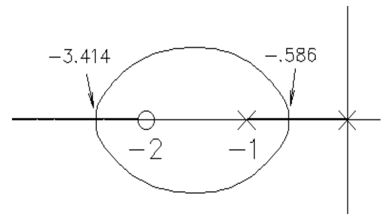
The breakaway points are also solutions to: $$\sum_{all} \frac{1}{(s + -p_i)} = \sum_{all} \frac{1}{(s + -z_i)}$$

IE: $$\frac{1}{(s + -p_1)} + \frac{1}{(s + -p_2)} + \frac{1}{(s + -p_3)} + \ldots = \frac{1}{(s + -z_1)} + \frac{1}{(s + -z_2)} + \frac{1}{(s + -z_3)} + \ldots$$

Not all of these solutions will actually be breakaway points, some may be eliminated by the real-axis rule, and some may be arrival points. Both of these methods can be messy and result in high-order equations to solve. The second example will show an iterative way to deal with the complexity.

Example 1 $G(s) = \frac{s+2}{s\cdot(s+1)}$ Solve: $\frac{1}{s} + \frac{1}{s+1} = \frac{1}{s+2}$

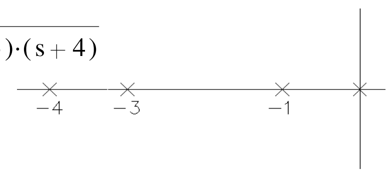
$$\frac{(s+1)+s}{s\cdot(s+1)} = \frac{1}{s+2}$$



$(2\cdot s+1)\cdot(s+2) = s\cdot(s+1)$ $s^2 + 4\cdot s + 2 = 0$ $s = -3.414$ $s = -0.586$

Example 2 Iterative process, best shown by example: $G(s) = \frac{1}{s\cdot(s+1)\cdot(s+3)\cdot(s+4)}$

Find the breakaway point between 0 and -1.



Must solve: $$\frac{1}{s} + \frac{1}{(s+1)} + \frac{1}{(s+3)} + \frac{1}{(s+4)} = 0$$

Guess $s = -0.4$ and use that for all the s's except those closest to the breakaway you want to find.

Solve this instead: $$\frac{1}{s} + \frac{1}{(s+1)} + \frac{1}{(-0.4+3)} + \frac{1}{(-0.4+4)} = 0$$

$$\frac{1}{s} + \frac{1}{(s+1)} + \left(\frac{1}{2.6} + \frac{1}{3.6}\right) = 0$$

multiply by $s(s + 1)$: $$\frac{s+1}{1} + \frac{s}{1} + s\cdot(s+1)\cdot\left(\frac{1}{2.6} + \frac{1}{3.6}\right) = 0$$

$$s^2 + 4.0194\cdot s + 1.5097 = 0$$

$$s = \frac{-4.0194 + \sqrt{4.0194^2 - 4\cdot 1.5097}}{2} = -0.419$$ Use this answer to try again

ignore the -3.6 solution for this answer.

$$\frac{1}{s} + \frac{1}{(s+1)} + \left(\frac{1}{2.581} + \frac{1}{3.581}\right) = 0$$

$$s^2 + 4\cdot s + 1.5 = 0$$

$$s = \frac{-4 + \sqrt{4^2 - 4\cdot 1.5}}{2} = -0.419$$ No significant change, so this is the breakaway point

To find the breakaway point between -3 and -4: Guess $s = -3.6$

$$\frac{1}{-3.6} + \frac{1}{(-3.6+1)} + \frac{1}{(s+3)} + \frac{1}{(s+4)} = 0$$

solve for s: $s = -3.58$ Not much change, so this is the breakaway point

Actually, it usually doesn't matter that much just where the breakaway points are.

8. Gain at any point on the root locus: $k = -\dfrac{1}{G(s)} = \dfrac{1}{|G(s)|} = \dfrac{|D(s)|}{|N(s)|}$

9. Phase angle of $G(s)$ at any point s on the root locus: $\arg(G(s)) = \arg(N(s)) - \arg(D(s)) = \pm 180^o \quad \pm 540^o \ldots$

Note: arg(x) is /(x)   Or: $\arg\left(\dfrac{1}{G(s)}\right) = \arg(D(s)) - \arg(N(s)) = \pm 180^o \quad \pm 540^o \ldots$

Or: $\arg(-G(s)) = 0^o \quad \pm 360^o \ldots$

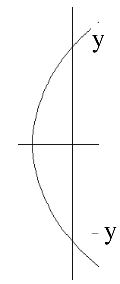
$$\sum_{\text{all zeroes}} (\text{angle of point } s \text{ relative to zero}) - \sum_{\text{all poles}} (\text{angle of point } s \text{ relative to pole}) = \pm 180^o \quad \pm 540^o \ldots$$

10. Gain at $j\omega$ crossing: Use Routh-Hurwitz test.

OR: a) Get a rough s (say y) value from your plot,

b) Check it (evaluate the angle of $G(jy)$) and iterate using rule 9,

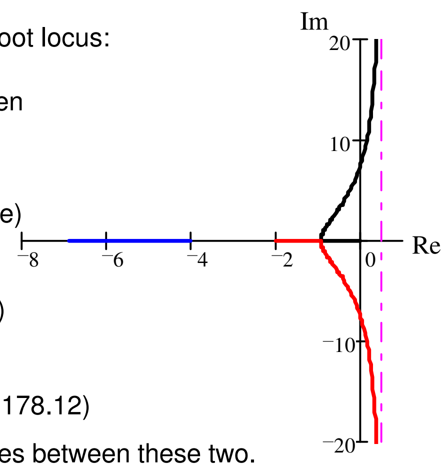
c) Find k using rule 8.



Calculator example: $G(s) = \dfrac{s+7}{s\cdot(s+2)\cdot(s+4)}$

Find the gain at $j\omega$ crossing:

Root locus:



Let's assume that the root locus crosses the $j\omega$ axis somewhere between 5 and 10. I first try 5, evaluating $\dfrac{1}{G(5j)}$ on my calculator

Note: I'm evaluating $1/G(s)$ so I'll end up with the gain value for free)

In a TI-86, I enter the following:

5.000->S:((0,S)*(2,S)*(4,S))/((7,S))   It returns: (20.04 /174.00)

Next I try:

10.00->S:((0,S)*(2,S)*(4,S))/((7,S))   TI returns: (89.98 /-178.12)

The first was a positive angle, and this is negative, yep, the answer lies between these two.

The first was 6^o under 180^o and the second is 2^o over, interpolate: $5 + \dfrac{6}{6+2}\cdot 5 = 8.75$

Try: 8.75   8.750->S:((0,S)*(2,S)*(4,S))/((7,S))   TI returns: (67.43 /-178.78)

$$8.75 - \frac{180 - 178.78}{180 - 178.12}\cdot(10 - 8.75) = 7.939$$

Try: 7.9   7.900->S:((0,S)*(2,S)*(4,S))/((7,S))   TI returns: (54.01 /-179.52)

$$7.9 - \frac{.48}{1.22}\cdot(8.75 - 7.9) = 7.566$$

Try: 7.5   7.500->S:((0,S)*(2,S)*(4,S))/((7,S))   TI returns: (48.23 /-179.97)

$$7.5 - \frac{.03}{.48}\cdot(7.9 - 7.5) = 7.475$$

Try: 7.475   7.475->S:((0,S)*(2,S)*(4,S))/((7,S))   TI returns: (47.88 /179.99)

The root locus crosses at $\pm 7.475j$ and the gain is 48. $k = 48$

11. Departure angle (θ_D) from a complex pole (p_c).

Recall rule 9 (one of the most important rules):

for any point s on the root locus:

$$\arg(G(s)) = \arg(N(s)) - \arg(D(s)) = \pm 180^o \quad \pm 360^o \ldots$$

Note: arg(x) is ∠(x)

For multiple (r) poles:

Divide the circle into r divisions: $\frac{360 \cdot deg}{r}$

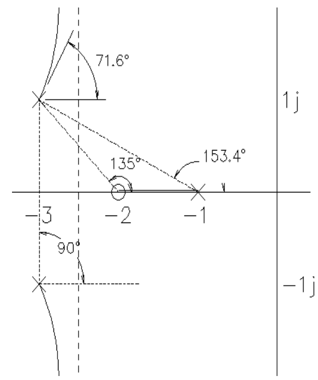
and rotate all by $\frac{\theta_D}{r}$

Now imagine a point ε-distance away from the complex pole. That point would have an angle of θ_D with respect to the complex pole, but it' s angle relative to all the other poles and zeros would be essentially the same as the complex pole.

$$\sum_{\text{all zeroes}} (\text{angle of point s relative to zero}) - \sum_{\text{all poles but } p_c} (\text{angle of point s relative to pole}) - \theta_D = \pm 180^o \quad \pm 540^o \ldots$$

Example: $G(s) := \frac{s+2}{(s+1)\cdot\left[(s+3)^2+1^2\right]}$

Find the departure angle from the pole at: $p_c := -3 + 1\cdot j$



$$135 - 153.4 - 90 - \theta_D = \pm 180^o \quad \pm 540^o \ldots$$

rearrange: $\theta_D = 180 - 90 - 153.4 + 135 = 71.6 \ deg$

Mathmatically: $\theta_D = 180\cdot deg + \arg\left[G(p_c)\cdot(s+p_c)\right]$

The O-L phase angle computed at the complex pole, but ignoring the effect of that complex pole.

Our example: $\theta_D = 180\cdot deg + \arg\left[\frac{p_c + 2}{(p_c+1)\cdot(p_c+3+1\cdot j)}\right] = 71.6 \cdot deg$

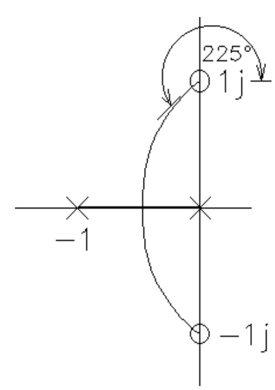
12. Arrival angle (θ_A) to complex zero (z_c).

Exactly the same idea.

$$\sum_{\text{all zeroes but } z_c} (\text{angle of point s relative to zero}) + \theta_A - \sum_{\text{all poles}} (\text{angle of point s relative to pole}) = \pm 180^o \quad \pm 540^o \ldots$$

Example: $G(s) := \frac{s^2+1^2}{s\cdot(s+1)} = \frac{(s-1\cdot j)\cdot(s+1\cdot j)}{s\cdot(s+1)}$

Find the departure angle from the pole at: $z_c := 1\cdot j$



$$90 + \theta_A - 90 - 45 = \pm 180^o \quad \pm 540^o \ldots$$

rearrange: $\theta_A = 180 - 90 + 90 + 45 \ deg25$

Mathmatically: $\theta_A = 180\cdot deg - \arg\left[\frac{G(z_c)}{(s+z_c)}\right]$

The O-L phase angle computed at the complex zero, but ignoring the effect of that complex zero.

Our example: $\theta_A = 180\cdot deg - \arg\left[\frac{1\cdot j + 1\cdot j}{1\cdot j\cdot(1\cdot j+1)}\right] = 225 \cdot deg$