<u>Example counting</u>: A president, vice-president and treasurer are to be chosen in a student club which has 40 members.

* How many choices of officers are possible?

$$_{40}P_3 = \frac{40!}{(40-3)!} = 40 \times 39 \times 38 = 59,280$$

We use permutation because who serves in which office matters.

* How many choices if A will only serve if he is president?

2 cases : Case 1 – A serves as president

$_{39}P_2$ choices for other two offices

Case 2 – A doesn't serve at all

$_{39}P_3$

$$\text{total \# choices} = {}_{39}P_2 + {}_{39}P_3 = 39 \times 38 + 39 \times 38 \times 37$$
$$= 1,482 + 54,834$$
$$= 56,316$$

* How many choices if A is not eligible to serve as president but can serve in other offices?

Case 1: A doesn't serve $_{39}P_3$

Case 2: A serves as vice-president $_{39}P_2$

Case 3: A serves as treasurer $_{39}P_2$

} Sum of these three

* How many choices if A and B will serve together or not at all.

Case 1: A & B don't serve ${}_{38}P_3$

Case 2: A & B serve -

Subcases	President	Vice-President	Treasurer
	A	B	X
	B	A	X
	A	X	B
	B	X	A
	X	A	B
	X	B	A

6 sub-cases

Could have found with $3! = 6$ because this is how many ways we can order A, B and X

For each of these there are ${}_{38}P_1 = 38$ remaining choices for the X position

Therefore 6×38

So total choices $= {}_{38}P_3 + 6 \times 38$

$= 38 \times 37 \times 36 + 6 \times 38$

$= 50{,}616 + 228$

$= 50{,}844$

<u>Example counting</u>: A deck of 13 cards contains one of each card 1 - 10 and Jack, Queen and King. If we draw a hand of 4 cards without replacement from this deck, what is the probability of getting a King and a Queen? Assume all outcomes are equally likely.

* First lets find # of all possible outcomes (order doesn't matter)

$$ {}_{13}C_4 = \frac{13!}{9!\,4!} = \frac{13 \times \cancel{12} \times 11 \times \cancel{10}^{5}}{\cancel{4 \times 3} \times \cancel{2} \times 1} = 715 $$

* Now find # outcomes with a K and Q (again order doesn't matter)

The K and Q take up 2 spots and there are 2 spots left. Also there are 11 cards left to choose from. So

$$ {}_{11}C_2 = \frac{11!}{9!\,2!} = \frac{11 \times 10}{2 \times 1} = 55 $$

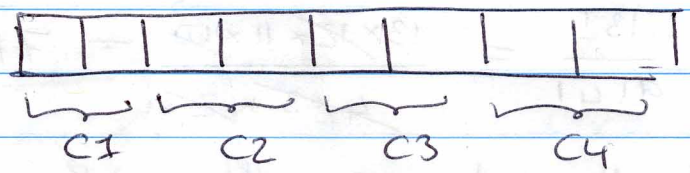
$$ P(K, Q \text{ in hand of } 4) = \frac{55}{715} = 0.077 $$

<u>Example counting</u>: Four married couples are going to sit in 8 seats in a row at a concert.

* How many ways can they sit if there are no restrictions?

$$8! = 40320$$

* If each couple must sit next to each other



Think of each couple as an unit. There are 4 units.

The 4 units can be organized in $4! = 24$ ways.

Now each couple could sit in 2 different ways Husband left, wife right or vice versa.

Then for any unit organization from above, we have 2^4 possibilities.

So answer is $24 \times 2^4 = 384$

<u>Example counting</u>: Given the digits 0, 1, 2, 3, 4, 5, 6 how many 4 digit numbers can be written? Note you can't use 0 in the thousands position for a four digit number.

<u>Step 1</u> $\underline{6} \times \underline{\quad} \times \underline{\quad} \times \underline{\quad}$ 6 possibilities for thousands position $\{1, 2, 3, 4, 5, 6\}$

<u>Step 2</u> $\underline{6} \times \underline{6} \times \underline{5} \times \underline{4}$ 6 remaining possibilities for hundreds position
5 for tens
4 for units

$= \underline{\underline{720}}$

(Alternative solution:)

possibilities if 0 can be used in the thousands position is ${}_7P_4$

Of these how many have 0 in the thousands position? Well, if 0 is used in the thousands position the rest is the same as selecting 3 digits out of the 6 so ${}_6P_3$

Therefore answer is ${}_7P_4 - {}_6P_3 = 7 \times 6 \times 5 \times 4 - 6 \times 5 \times 4$

$= 840 - 120 = \underline{\underline{720}}$

<u>Example</u> : Lottery 60 participants 3 prizes: car, laptop and \$5. What is the probability Joe wins the car, Brian wins the laptop and Kate wins \$5?

soln. Who wins what (order) matters, so total # outcomes ${}_{60}P_3$. The event includes only 1 outcome; therefore the answer is $1/{}_{60}P_3 = \frac{1}{60 \times 59 \times 58}$

* What is the probability Joe wins the car?

soln. # outcomes with Joe winning the car is ${}_{59}P_2$

This is because if Joe wins the car the other 2 prizes will go to two of the 59 remaining people.

therefore $P(\text{Joe wins car}) = \frac{{}_{59}P_2}{{}_{60}P_3}$

$$= \frac{\cancel{59 \times 58}}{60 \times \cancel{59 \times 58}} = 1/60$$

which is probably what you'd have guessed intuitively

* Now lets say all three prizes are the same, i.e. a laptop. So order doesn't matter. What is the probability that Joe, Brian & Kate win?

soln. # outcomes with Joe, Brian & Kate winning disregarding order is 1

total outcomes disregarding order is ${}_{60}C_3$

so the prob is $1/{}_{60}C_3 = \frac{3 \times 2 \times 1}{60 \times 59 \times 58}$

Example: Dice game (Yahtzee). 5 dice are rolled. To win you must get three of a kind and two of a kind (not 5 of the same kind though). Find the probability of winning.

Soln. (1) Total # outcomes $6 \times 6 \times 6 \times 6 \times 6 = 6^5$
5 dice, each six faces

(2) Total # winning outcomes

example winning outcome: $\boxed{4}$ $\boxed{2}$ $\boxed{4}$ $\boxed{2}$ $\boxed{2}$
red green blue yellow white

Imagine the dice have colors, so we can keep track of the combinations.

2a) In how many ways can I choose 3 dice out of 5?

2b) In how many ways can I choose the remaining 2 dice?

2c) For the triplet, how many possible values?

2d) " " double, " " " "

$2a \times 2b \times 2c \times 2d = \#$ winning outcomes

2a) ${}_5C_3 = 10$ 2b) ${}_2C_2 = 1$

2c) The triplet in 2a could have been 1s, 2s, 3s, 4s 5s or 6s. So 6 ways

2d) 5 ways to avoid having 5 dice of the same kind

$10 \times 1 \times 6 \times 5 = 300$ winning outcomes

$P(\text{win}) = \frac{300}{6^5} = 0.0386 \approx 1/26$