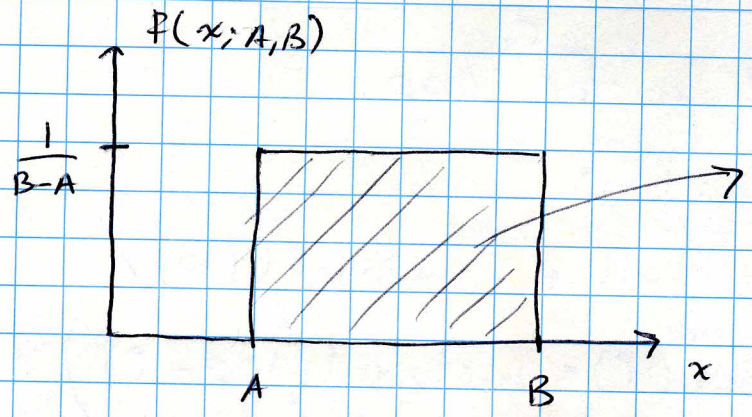
CONTINUOUS UNIFORM DISTRIBUTION

<u>Defn</u> : The probability density function of the continuous uniform random variable X on the interval $[A,B]$ is

$$f(x;A,B) = \begin{cases} \dfrac{1}{B-A}, & A \le x \le B \\ 0, & \text{elsewhere} \end{cases}$$



Mean of $f(x;A,B)$:

$$\mu = \int_{-\infty}^{\infty} x f(x;A,B)\,dx = \int_A^B x \frac{1}{B-A}\,dx = \frac{x^2}{2(B-A)}\Bigg|_A^B$$

$$= \frac{B^2}{2(B-A)} - \frac{A^2}{2(B-A)} = \frac{B^2-A^2}{2(B-A)} = \frac{\cancel{(B-A)}(B+A)}{2\cancel{(B-A)}}$$

$= (B+A)/2$ at the center of the range $[A,B]$ as we would expect.

Variance of $f(x;A,B)$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x;A,B)\,dx - \mu^2$$

$$= \int_A^B \frac{x^2}{B-A}\,dx - \left(\frac{B+A}{2}\right)^2$$

$$= \frac{x^3}{3(B-A)}\Bigg|_A^B - \frac{(B+A)^2}{4}$$

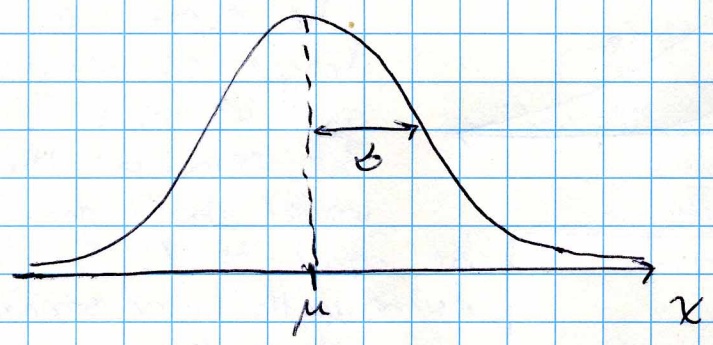
$$= \frac{B^3 - A^3}{3(B-A)} - \frac{(B+A)^2}{4}$$

$$= \frac{\cancel{(B-A)}(B^2+AB+A^2)}{3\cancel{(B-A)}} - \frac{(B+A)^2}{4}$$

$$= \frac{4B^2+4AB+4A^2-3B^2-6AB-3A^2}{12}$$

$$= \frac{B^2-2AB+A^2}{12} = \frac{(B-A)^2}{12}$$

CONTINUOUS NORMAL DISTRIBUTION (a.k.a. Gaussian)



Many physical phenomena are described very well by a normal distribution:

- Human height
- Measurement errors
- Daily changes in Dow Jones index
- many more

Described by 2 parameters mean μ and standard deviation σ

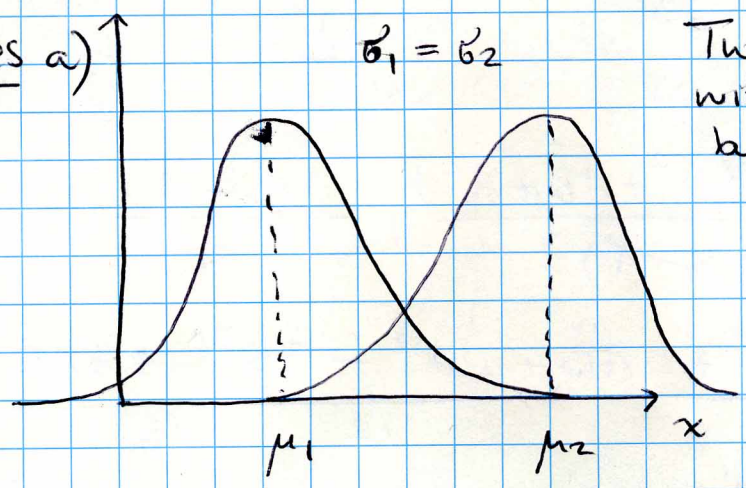
<u>Defn</u>: The prob. density function of the normal (Gaussian) random variable X with mean μ and standard deviation σ is

$$n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

<u>Notes</u>:
- $(x-\mu)^2$ is squared distance from the mean
- $e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ gets smaller as $(x-\mu)^2$ gets larger

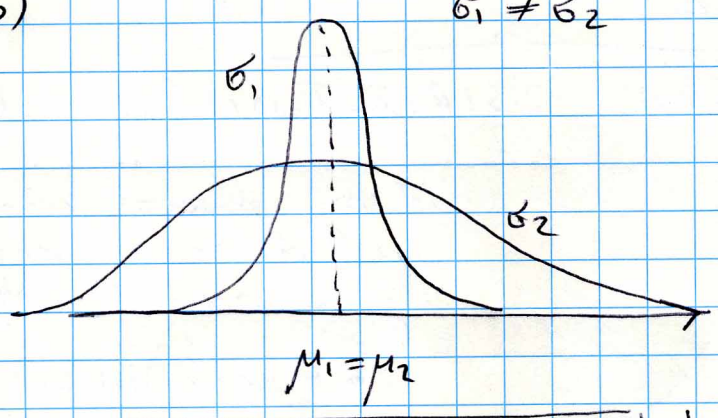
  How fast it gets small depends on σ. Faster for small σ.
- The term $\frac{1}{\sqrt{2\pi}\,\sigma}$ makes sure $\int_{-\infty}^{\infty} n(x;\mu,\sigma)\,dx = \underline{1}$

Examples a)



Two normal distributions with the same variance but different means.

b)



Two normal distributions with the same mean but different variances

The curve with the larger standard dev. σ_2 spreads further. But remember the area under both needs to be equal to 1. So the curve with lower standard dev. σ_1 is higher at its peak.

Observations

- The mode of a density function is the x value for which $f(x)$ is highest. For the normal density the mode is equal to the mean.
- The normal distribution is symmetric about μ.
- The normal curve has points of inflection at $\mu \pm \sigma$
- The " " approaches 0 asymptotically as we move away from μ in either direction.

Question = How do we know that the mean of $n(x; \mu, \sigma)$ is really μ and the variance is really σ^2?

<u>Proof of mean</u>

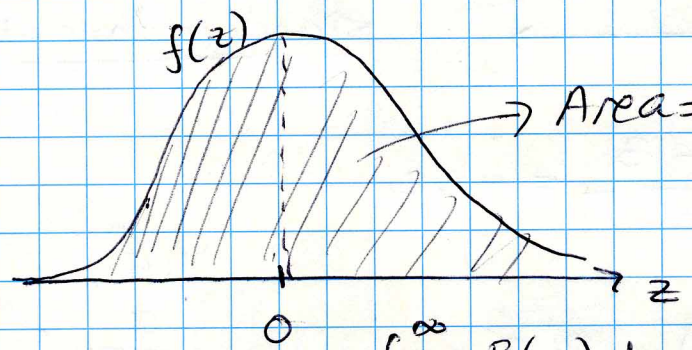
$$\mu = E[X] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

let $z = \dfrac{x-\mu}{\sigma}$ and $dx = \sigma\, dz$

then $$\mu = E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z)\, e^{-\frac{z^2}{2}}\, dz$$

$$E[X] = \mu \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz}_{\text{This is the entire area under the normal curve. Hence } = 1} + \sigma \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{z^2}{2}}\, dz}_{\text{evaluates to 0 because of symmetry.}}$$

$$= \mu \cdot 1 + \sigma \cdot 0 = \mu.$$



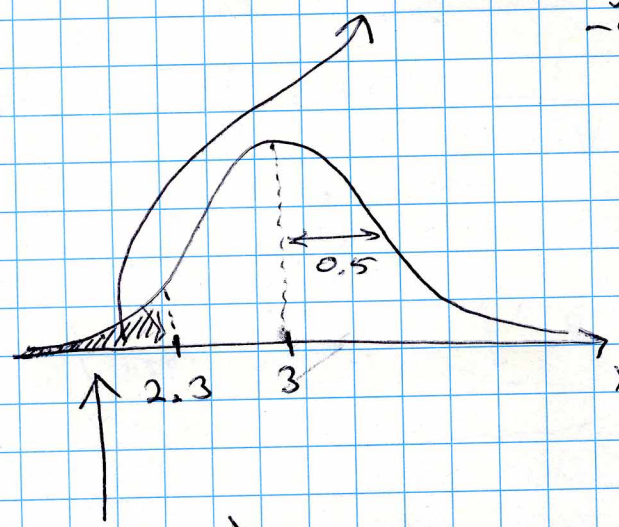
$$\int_{-\infty}^{\infty} z f(z)\, dz = 0 \quad \text{due to symmetry}$$

The proof of variance $= \sigma^2$ can be accomplished with a similar change of variables followed by integration by parts.

See page 175 of textbook.

<u>Example</u>: A certain type of battery lasts, on average 3 years with a standard deviation of 0.5 years. Assuming battery lives are normally distributed, find the probability that a given battery will last less than 2.3 years.

<u>Soln</u>: $$P(X<2.3) = \int_{-\infty}^{2.3} \frac{1}{\sqrt{2\pi}\,0.5} e^{-\frac{(x-3)^2}{2\times 0.5^2}}\,dx$$

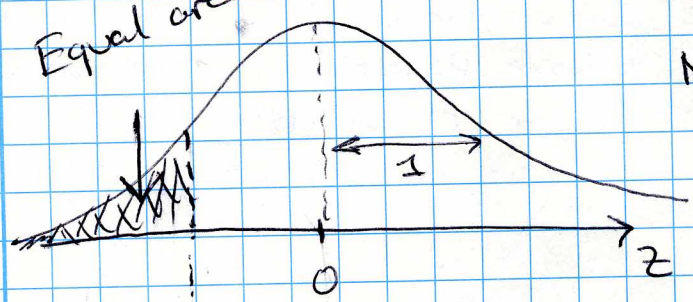


Unfortunately, the difficulty encountered in solving integrals of the normal density function necessitates tabulation of normal curve areas for reference. But can't make tables for all possible μ, σ!

Let $Z = \dfrac{X-\mu}{\sigma}$

Notice Z has 0 mean and variance $=1$. It is said to have a <u>standard normal dist.</u>

Equal areas



$$z = \frac{2.3-3}{0.5} = -1.4$$

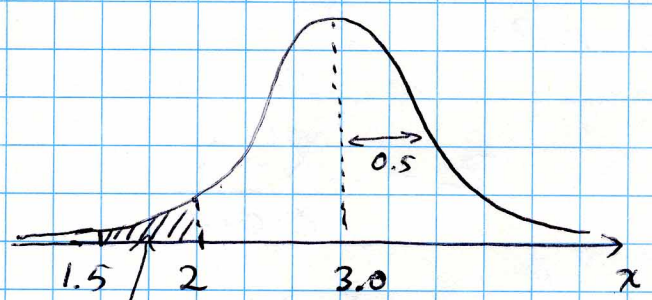
Now $$P(X<2.3) = P(Z<-1.4) = \int_{-\infty}^{1.4} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

So we only need table for the standard normal distribution to be able to evaluate integrals of arbitrary normal distributions with any μ, σ

$\boxed{\text{Table A.3 textbook}}$

$$P(Z<-1.4) = 0.0808 \implies P(X<2.3) = 0.0808$$

<u>Example</u>: For the same battery life problem, what is the probability that a given battery will last between 1.5 to 2 years?



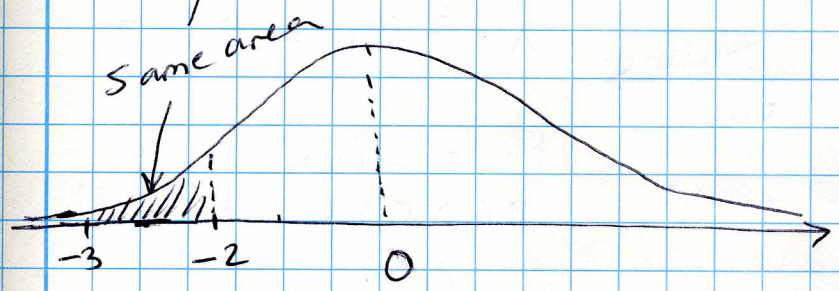
$$z = \frac{X-3}{0.5}$$

- when $x = 2$

$$z = \frac{2-3}{0.5} = -2$$

- when $x = 1.5$

$$z = \frac{1.5-3}{0.5} = -3$$



$$P(1.5 < X < 2) = P(-3 < z < -2)$$
$$= P(z < -2) - P(z < -3)$$

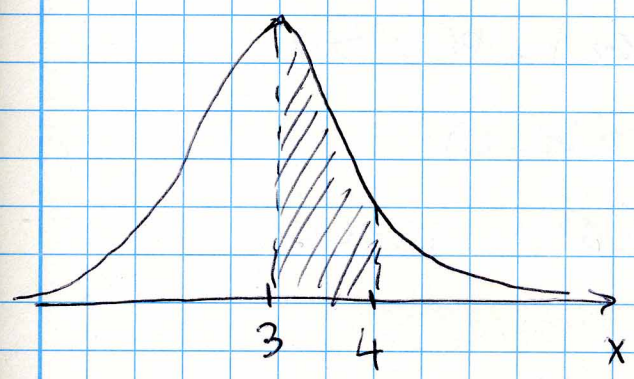
From Table A.3.

$$P(z < -2) = 0.0228 \qquad P(z < -3) = 0.0013$$

Therefore

$$P(1.5 < X < 2) = 0.0228 - 0.0013 = 0.0215$$

$$P(3 < X < 4) = ? \qquad z = \frac{X-3}{0.5}$$



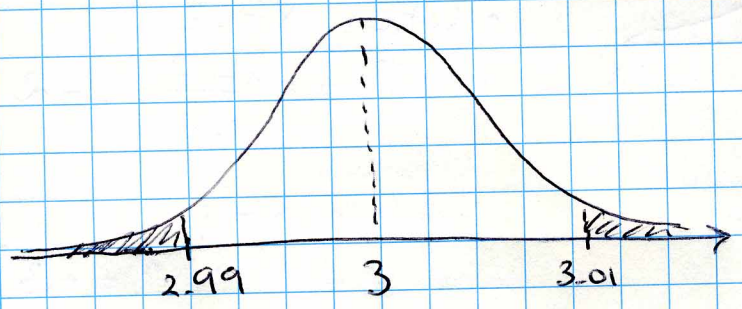
$$x = 3 \longrightarrow z = 0$$
$$x = 4 \longrightarrow z = 2$$

$$P(3 < X < 4) = P(z < 2) - P(z < 0)$$
$$= 0.9772 - 0.5$$
$$= 0.4772$$

<u>Example</u>: A company manufactures electrical resistors with mean 3 Ohms and standard deviation 0.005 Ohms. A buyer sets specifications on the resistance to be 3.0 ± 0.01 Ohms. On the average, what percentage of resistors will be rejected?

<u>Soln</u>:

$$Z = \frac{X-3}{0.005}$$



$X = 2.99 \rightarrow Z = -2$

$X = 3.01 \rightarrow Z = 2$

First find $P(2.99 < X < 3.01)$

Sum of the 2 shaded areas will be rejected.

$$P(2.99 < X < 3.01) = P(-2 < Z < 2)$$
$$= P(Z < 2) - P(Z < -2) = 0.9772 - 0.0228$$
$$= 0.9544$$

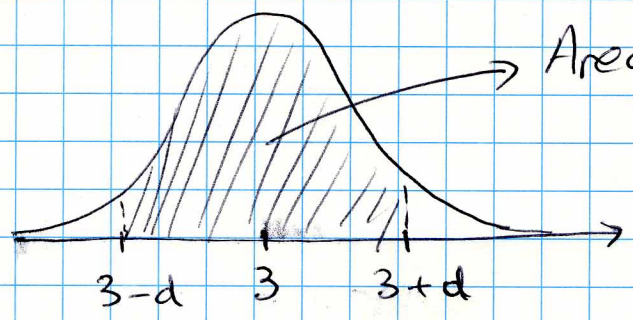
Then $P(\text{reject}) = P(Z < -2) + P(Z > 2)$
$$= 1 - P(-2 < Z < 2)$$
$$= 1 - 0.9544 = 0.0456$$

<u>Alternative soln</u>. Due to symmetry
$$P(Z < -2) = P(Z > 2)$$

Hence $P(\text{reject}) = P(Z < -2) + P(Z > 2)$
$$= 2 \cdot P(Z < -2)$$
$$= 2 \times 0.0228 = 0.0456$$

<u>Example</u>: Using the Normal curve in reverse

A company manufactures electrical resistors with mean 3 Ohms. The standard deviation is 0.1 Ohms. Find a value d such that 95% of all manufactured resistors fall in the range 3 ± d Ohms.

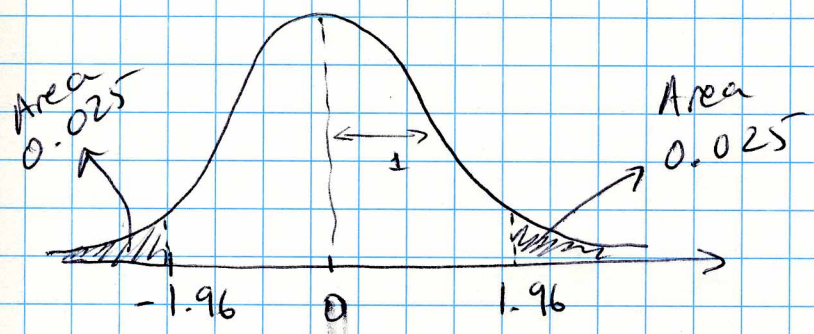


Area = 0.95

$$Z = \frac{X-3}{0.1}$$

From Table A.3 we know that $P(Z < -1.96) = 0.025$

Hence $P(-1.96 < Z < 1.96) = 1 - 2 \times 0.025$
$= 0.95$



Finally we have to convert $z = 1.96$ back to X

$$Z = \frac{X-3}{0.1} \quad \text{so} \quad X = 0.1Z + 3$$

$0.1 \times 1.96 + 3 = 3.196$
$0.1 \times -1.96 + 3 = 2.804$

$P(2.804 < X < 3.196) = 0.95$

$3 \pm 0.196 \implies d = 0.196$