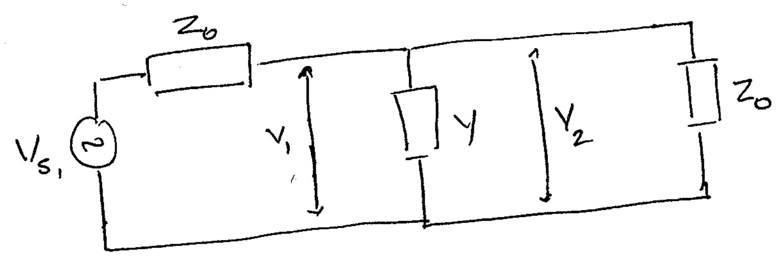
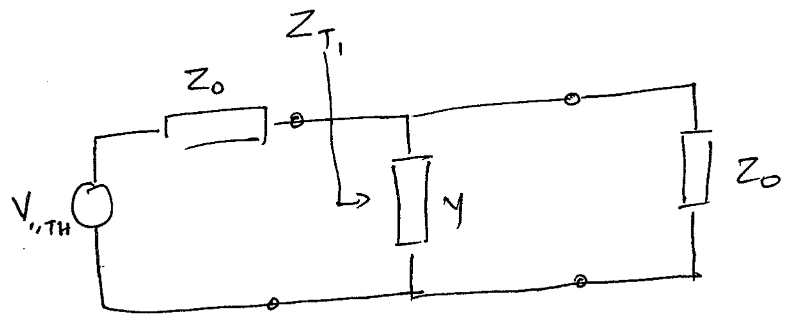
# Lecture 3 notes

example 2:-

Find the S parameters of a shunt admittance Y connected between the two ports




The Thevenin's equivalent circuit can be drawn as:




$$S_{11} = \frac{Z_{T_1} - Z_0}{Z_{T_1} + Z_0} =$$

$$Z_{T_1} = \frac{1}{Y} \parallel Z_0 = \frac{Z_0 \frac{1}{Y}}{\frac{1}{Y} + Z_0} = \frac{Z_0}{1 + Z_0 Y}$$

$$S_{11} = \frac{\frac{Z_0}{1+Z_0 Y} - Z_0}{\frac{Z_0}{1+Z_0 Y} + Z_0} = \frac{Z_0 - Z_0 - Z_0^2 Y}{Z_0 + Z_0 + Z_0^2 Y} = -\frac{Z_0 Y}{2 + Z_0 Y}$$

$$S_{22} = \Gamma_2 \Big|_{\Gamma_S = 0} = \frac{Z_2 - Z_{T2}}{Z_2 + Z_{T2}} = \frac{-Z_0 Y}{2 + Z_0 Y}$$

$$V_2 = V_{S_1} \frac{Z_{T_1}}{Z_{T_1} + Z_0} = \frac{V_{S_1}}{2 + Z_0 Y}$$

$$S_{21} = \frac{2V_2}{V_{S_1}} \Bigg|_{\Gamma_L = 0} = \frac{2}{2 + Z_0 Y}$$

$$S_{12} = \frac{2V_1}{V_{S_2}} \Bigg|_{\Gamma_S = 0} = \frac{2}{2 + Z_0 Y}$$

$$= \frac{2 + Z_0 Y - Z_0 Y}{2 + Z_0 Y}$$

$$= 1 - \frac{Z_0 Y}{2 + Z_0 Y}$$

$$= 1 - \Gamma_1$$

$$|S_{11}|^2 + |S_{21}|^2 = 1$$

$$S_{11} S_{12}^* + S_{21} S_{22}^* = 0$$

For the 2 port N/w shown in the figure

1) Find $Z_{IN}(0)$

2) Evaluate $a_1(0)$, $b_1(0)$, $a_1(\frac{\lambda}{8})$, $b_1(\frac{\lambda}{8})$ and $a_2(0)$

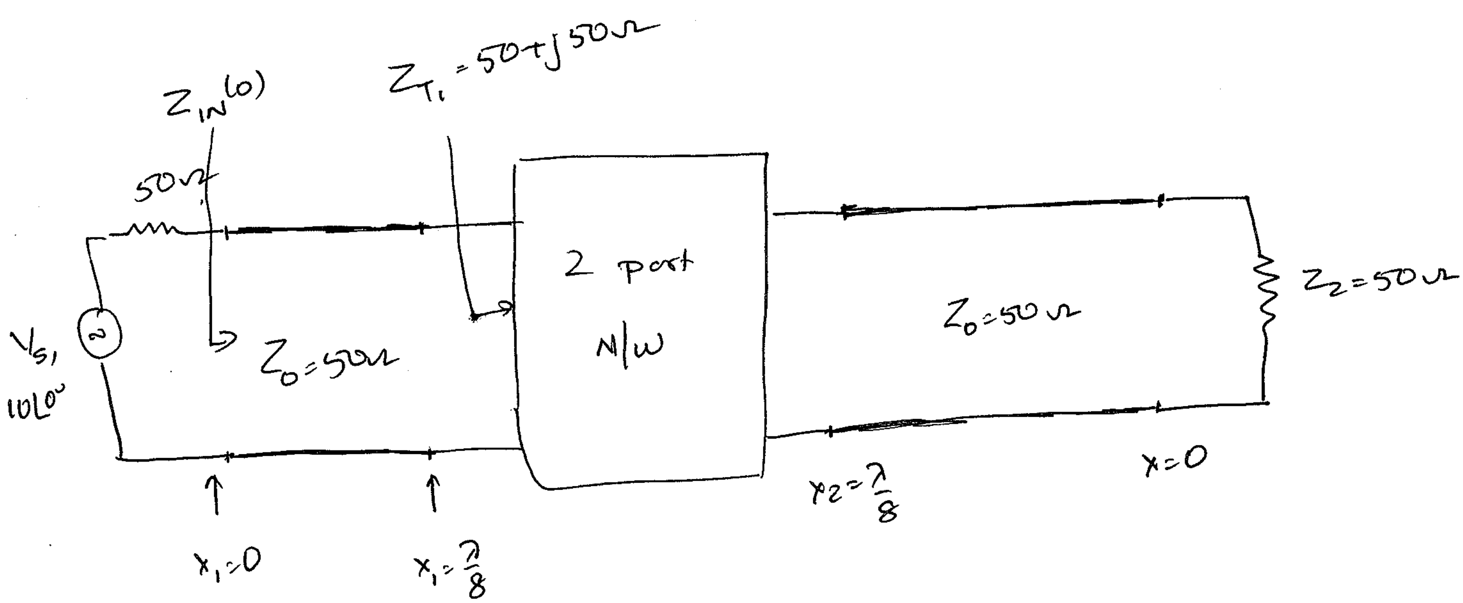
3) Evaluate $V_1(0)$, $V_1(\frac{\lambda}{8})$, $I_1(0)$ and $I_1(\frac{\lambda}{8})$

4) Evaluate the average input power at $x_1 = 0$ and at $x_1 = \frac{\lambda}{8}$

5) Evaluate $S_{11}$ at $x_1 = 0$ and $x_1 = \frac{\lambda}{8}$

6) Evaluate i/p VSWR and o/p VSWR

7) If scattering parameters measured at $x_1 = x_2 = \frac{\lambda}{8}$ reference planes are $S_{11} = 0.447\angle 63.4^\circ$, $S_{12} = 0.01\angle 40^\circ$ $S_{21} = 5\angle 135^\circ$ and $S_{22} = 0.6\angle 40^\circ$ calculate power delivered to load $Z_2$



a)

$$Z_{IN} = Z_0 \frac{Z_L + j Z_0 \tan\beta d}{Z_0 + j Z_L \tan\beta d}$$

$$\beta d = \frac{2\pi}{\lambda} \times \frac{\lambda}{8} = \frac{\pi}{4}$$

$$Z_{IN} = 100 - j50\ \Omega$$

b) From page 13 of lecture 1 notes and 1.6.3 from textbook

$$a_1(0) = \frac{1}{2\sqrt{Z_0}} \left[V_1(0) + Z_0 I_1(0)\right]$$

$$= \frac{1}{2\sqrt{50}} \left[\underbrace{10 - 50 I_1(0)}_{V_1(0)} + 50 I_1(0)\right]$$

$$= \frac{1}{\sqrt{2}}$$

$$l_1 = \frac{\lambda}{8} \quad \theta_1 = \beta l_1 = \frac{\pi}{4}$$

$$a_1\left(\frac{\lambda}{8}\right) = a_1(0)\, e^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \angle 45^\circ$$

From page 13 of lecture 1 notes and 1.6.4

$$b_1(0) = \frac{1}{2\sqrt{Z_0}} \left[V_1(0) - Z_0 I_1(0)\right] = \frac{1}{2\sqrt{50}} \left[10 - 2(50) I_1(0)\right]$$

$$I_1(0) = \frac{V_s}{50+Z_{IN}(0)} = \frac{10}{50+100-j50} = 0.063\angle 18.435^\circ$$

$$b_1(0) = 0.316\angle -26.57^\circ$$

$$b_1\left(\frac{\lambda}{8}\right) = b_1(0)\,e^{j\pi/4} = 0.316\angle -26.57^\circ(1\angle 45^\circ) = 0.316\angle 18.48^\circ$$

At o/p port the line is matched $\therefore\ a_2(0) = 0$

c)

$$V_1(0) = \frac{V_{s_1} Z_{IN}(0)}{50+Z_{IN}(0)} = 7.07\angle -8.13^\circ$$

$$V_1\left(\frac{\lambda}{8}\right) = \sqrt{Z_0}\left[a_1\left(\frac{\lambda}{8}\right) + b_1\left(\frac{\lambda}{8}\right)\right]$$

$$= 6.32\angle -26.57^\circ$$

$$I_1\left(\frac{\lambda}{8}\right) = \frac{V_1\left(\frac{\lambda}{8}\right)}{Z_{T_1}} = 0.089\angle -71.57^\circ\,A$$

d) $$P_1(0) = \frac{1}{2}\,\mathrm{Re}\left[V_1(0)\,I_1^*(0)\right] = 0.2W$$

$$P_1\left(\frac{\lambda}{8}\right) = \frac{1}{2}\,\mathrm{Re}\left[V_1\left(\frac{\lambda}{8}\right) I_1^*\left(\frac{\lambda}{8}\right)\right] = 0.2W$$

We can also calculate the power by using the formula

$$P_{AVS} = \frac{1}{2} |a_1(0)|^2 = 0.25W$$

from 1.6.15

$$P_1(0) = P_1\left(\frac{\lambda}{8}\right) = P_{avs}\left(1 - |S_{11}|^2\right) = 0.25\left(1 - (0.447)^2\right) = 0.2W$$

e)

$$S_{11} = \frac{Z_{T_1} - Z_0}{Z_{T_1} + Z_0} = \frac{(50 + j50) - 50}{(50 + j50) + 50} = 0.447 \angle 63.43^\circ$$

$S_{11}$ at $x_1 = 0$ can be obtained as

$$S_{11}' = \frac{Z_{IN}(0) - Z_0}{Z_{IN}(0) + Z_0} = \frac{(100 - j50) - 50}{(100 - j50) + 50} = 0.447 \angle -26.57^\circ$$

f)

o/p VSWR = 1

i/p VSWR $= \frac{1 + |S_{11}|}{1 - |S_{11}|} = 2.62$

g) Since o/p is matched $a_2(0) = a_2\left(\frac{\lambda}{8}\right) = 0$

$$b_2\left(\frac{\lambda}{8}\right) = S_{21} a_1\left(\frac{\lambda}{8}\right)$$

$$b_2\left(\frac{\lambda}{8}\right) = 5\angle 135^\circ \left(\frac{1}{\sqrt{2}}\angle -45^\circ\right) = 3.54\angle 90^\circ$$

$$b_2(0) = b_2\left(\frac{\lambda}{8}\right) e^{-j\frac{\pi}{4}} = 3.54\angle 45^\circ$$

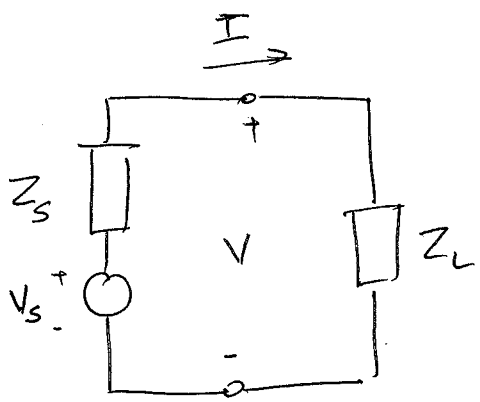
Power delivered to $Z_2$ is

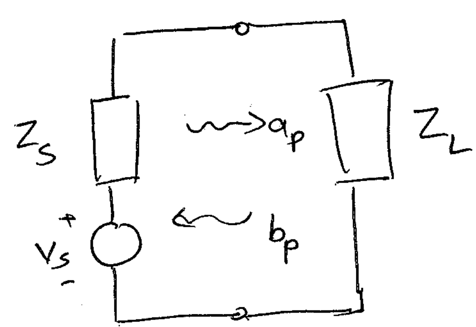
$$P_2(0) = \frac{1}{2}\left|b_2(0)\right|^2 = 6.27\,W.$$

## Power Waves and Generalized Scattering Parameters

We can use power waves in the presence of lumped element circuits.

Consider a circuit as shown in the figure below:



For this circuitry traveling wave analysis will not be suitable. Since there is no transmission line, and therefore the characteristic impedance is not defined.

In addition, load reflection coefficient has no meaning. For analyzing these circuits we can introduce a new set of waves called power waves. The power waves $a_p$ and $b_p$ are defined as.

$$a_p = \frac{1}{2\sqrt{R_s}} (V + Z_s I) \qquad \text{—} \quad (1)$$

$$b_p = \frac{1}{2\sqrt{R_s}} (V - Z_s^* I) \qquad \text{—} \quad (2)$$

where $R_s = Re [Z_s]$

These definitions are given such that $\frac{1}{2} |a_p|^2$ is equal to the power available from the source, and the reflected powerwave $b_p = 0$ when load impedance is conjugately matched to the source impedance.

(ie when $Z_L = Z_s^*$)

From the figure in page 7 we can write the voltage 'V' as

$$V = V_s - Z_s I \qquad - (3)$$

Substituting (3) in (1) we get

$$a_p = \frac{1}{2\sqrt{R_s}} (E_s - Z_s I + Z_s I) = \frac{V_s}{2\sqrt{R_s}}$$

Hence

$$|a_p|^2 = \frac{|V_s|^2}{4R_s} \qquad — \quad (4)$$

or

$$\frac{1}{2}|a_p|^2 = |a_{p,rms}|^2 = \frac{|E_s|^2}{8R_s} \qquad — \quad (5)$$

which is power available from source (ie $P_{AVS} = \frac{1}{2}|a_p|^2$

Maximum power will be delivered to the load when $Z_L = Z_S^*$

Power delivered to the load is

$$P_L = \frac{1}{2}|I|^2 Re(Z_L) = \frac{1}{2}\left[\frac{V_s}{Z_S + Z_L}\right]^2 Re[Z_L]$$

which will attain its maximum when $Z_L = Z_S^*$

$$P_{L,max} = P_{avs} = \frac{1}{8}\frac{|V_s|^2}{R_s} \quad — (6)$$

$$\frac{1}{2}\left[\frac{V_s^2}{(R_s + jX_s) + (R_s - jX_s)}\right]^2 R_s$$

$$\frac{1}{2}\frac{V_s^2}{4R_s^2} R_s$$

$$= \frac{1}{8}\frac{|V_s|^2}{R_s}$$

Since $\frac{1}{2}|a_p|^2$ represents power available from the source, $\frac{1}{2}|b_p|^2$ should represent the reflected power.

$$\frac{1}{2}|a_p|^2 - \frac{1}{2}|b_p|^2 = \frac{1}{2} Re[V^* I]$$

which is the power dissipated in $Z_L$ (ie $P_L$)

$$P_L = \frac{1}{2}|a_p|^2 - \frac{1}{2}|b_p|^2 \qquad — \quad (7)$$

$$\frac{1}{2}|b_p|^2 = P_{AVS} - P_L \qquad — \quad (8)$$

Reflected power = Available power from source - power dissipated in Load.

Under conjugate match condition $\frac{1}{2}|b_p|^2 = 0$

A power wave reflection coefficient (also called as the generalized reflection coefficient) can be defined as the ratio of $b_p$ to $a_p$

$$\Gamma_p = \frac{b_p}{a_p} = \frac{V - Z_s^* I}{V + Z_s I} = \frac{Z_L - Z_s^*}{Z_L + Z_s^*} \qquad — \quad (9)$$

The power dissipated in the load can then be written as,

$$P_L = \frac{1}{2}|a_p|^2(1 - |\Gamma_p|^2) = P_{AVS}(1 - |\Gamma_p|^2) \qquad -(10)$$

From eqn (1) and (2) we can write the current & voltage as,

$$V = \frac{1}{\sqrt{R_s}}(Z_s^* a_p + Z_s b_p) \qquad -(11)$$

$$I = \frac{1}{\sqrt{R_s}}(a_p - b_p) \qquad -(12)$$

From (11) and (12) we can also define incident and reflected voltages and currents and relate them to power waves

Let

$$V = V_p^+ + V_p^- \quad \& \quad I = I_p^+ + I_p^-$$

$$V_p^+ = \frac{Z_s^*}{\sqrt{R_s}} a_p \qquad \text{— (13)}$$

$$V_p^- = \frac{Z_s}{\sqrt{R_s}} b_p$$

$$I_p^+ = \frac{a_p}{\sqrt{R_s}} = \frac{V_p^+}{Z_s^*}$$

$$I_p^- = \frac{b_p}{\sqrt{R_s}} = \frac{V_p^-}{Z_s}$$

The voltage reflection coefficient can be defined as.

$$\Gamma_V = \frac{V_p^-}{V_p^+} = \frac{Z_s b_p}{Z_s^* a_p} = \frac{Z_s}{Z_s^*} \Gamma_p = \frac{Z_s}{Z_s^*} \frac{Z_L - Z_s^*}{Z_L + Z_s}$$

$$\Gamma_I = \frac{I_p^-}{I_p^+} = \frac{b_p}{a_p} = \Gamma_p$$

For a two-port Network we can define generalized scattering parameters denoted by $S_{P11}, S_{P12}, S_{P21}, S_{P22}$ in terms of power waves as:

$$b_{P_1} = S_{P_{11}} a_{P_1} + S_{P_{12}} a_{P_2}$$

$$b_{P_2} = S_{P_{21}} a_{P_1} + S_{P_{22}} a_{P_2}$$

where

$$a_{P_1} = \frac{1}{2\sqrt{R_1}} (V_1 + Z_1 I_1)$$

$$a_{P_2} = \frac{1}{2\sqrt{R_2}} (V_2 + Z_2 I_2)$$

$$b_{P_1} = \frac{1}{2\sqrt{R_1}} (V_1 - Z_1^* I_1)$$

$$b_{P_2} = \frac{1}{2\sqrt{R_2}} (V_2 - Z_2^* I_2)$$

$$R_1 = Re[Z_1] \;\&\; R_2 = Re[Z_2]$$

The values of $S_P$ parameters depend on the terminal impedances $Z_1$ and $Z_2$, These impedances are called the reference impedances.

There is no way of directly measuring the generalized S parameters. For example

$$S_{p_{11}} = \left.\frac{b_{p_1}}{a_{p_1}}\right|_{a_{p_2}=0}$$

$S_{p_{11}}$ can be determined if $a_{p_1}$ and $b_{p_1}$ can be measured with $a_{p_2}=0$.

$$S_{p_{11}} = \frac{Z_{T_1} - Z_1^*}{Z_{T_1} + Z_1}$$

i/p power of 2 port N/w can be expressed as

$$P_{IN} = \frac{1}{2}|a_{p_1}|^2 - \frac{1}{2}|b_{p_1}|^2 = P_{AVS}(1 - |S_{p_{11}}|^2)$$

Although $S_p$ parameters cannot be measured, the S parameters of the two-port can be easily measured and $S_p$ parameters can be calculated in terms of S parameters.

# Two Port N/w Parameter Conversion

At a given frequency a two-port N/w can be described in terms of several parameters. Therefore it is desirable to have relations to convert from one set to another.

$$[V] = [z][I]$$

$$[V] = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad [I] = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$[z] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$[V^+] + [V^-] = [z]\left([I^+] - [I^-]\right)$$

or

$$\left([z] + [Z_0]\right)[I^-] = \left([z] - [Z_0]\right)[I^+]$$

where $Z_0$ is assumed to be real

$$[Z_0] = \begin{bmatrix} Z_0 & 0 \\ 0 & Z_0 \end{bmatrix}$$

∴ The scattering matrix in terms of Z parameters is given by

$$[S] = \frac{[b]}{[a]} = \frac{[V^-]}{[V^+]} = \frac{[I^-]}{[I^+]} = \left([z] + [Z_0]\right)^{-1}\left([z] - [Z_0]\right)$$

Solving for [z] we obtain

$$[z] = [Z_0]\left([I] + [S]\right)\left([I] - [S]\right)^{-1}$$

unit diagonal matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$