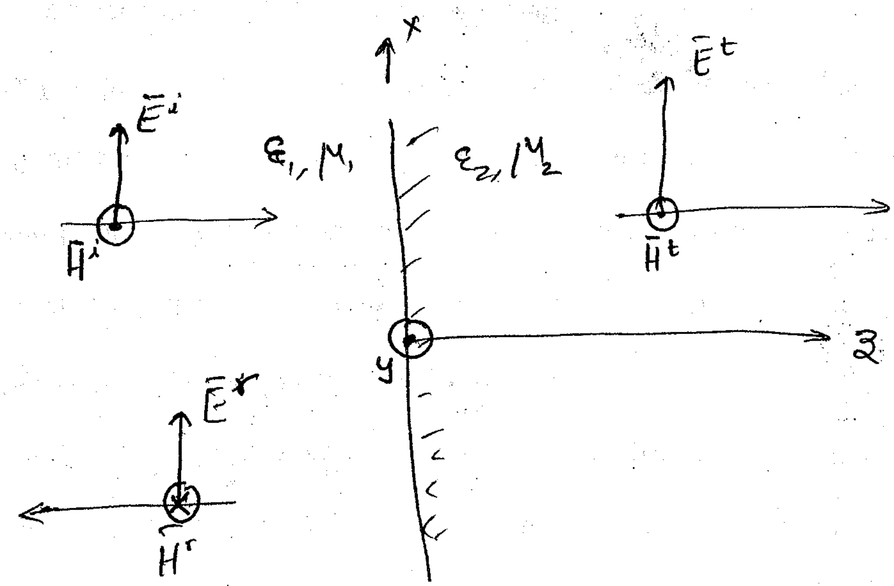
Chapter 5

Reflection and Transmission

Revision

In 3300 you observed the following

→ Normal incidence



$$\bar{E}^i = \hat{a}_x E_o e^{-j\beta_1 z}$$

$$\bar{E}^r = \hat{a}_x \Gamma^b E_o e^{j\beta_1 z}$$

$$\bar{E}^t = \hat{a}_x T^b E_o e^{-j\beta_2 z}$$

Assuming the incident electric field of amplitude E_o is polarized in the x direction, the expression for the incident, reflected and transmitted electric field components

Γ^b & T^b are used to represent the reflection & transmission coefficients at the interface. Since the incident fields are linearly polarized and the reflecting surface is planar, the reflected & transmitted fields will also be linearly polarized.

The corresponding magnetic fields are written as:

$$\bar{H}^i = \hat{a}_y \frac{E_o}{\eta_1} e^{-j\beta_1 z}$$

$$\bar{H}^r = -\hat{a}_y \frac{\Gamma^b E_o}{\eta_1} e^{j\beta_1 z}$$

$$\bar{H}^t = \hat{a}_y \frac{T^b E_o}{\eta_2} e^{-j\beta_2 z}$$

$$\Gamma^b = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\bar{E}^r}{\bar{E}^i} = -\frac{\bar{H}^r}{\bar{H}^i}$$

$$T^b = \frac{2\eta_2}{\eta_1 + \eta_2} = 1 + \Gamma^b = \frac{\bar{E}^t}{\bar{E}^i} = \frac{\eta_2}{\eta_1} \frac{\bar{H}^t}{\bar{H}^i}$$

Away from the interface the reflection Γ and transmission T coefficients are related to those at the boundary, and can be written as

$$\Gamma(z=-l_1) = \frac{\bar{E}^r(z)}{\bar{E}^i(z)}\Bigg|_{z=-l} = \frac{\Gamma^b E_o e^{j\beta_1 z}}{E_o e^{-j\beta_1 z}}\Bigg|_{z=-l} = \Gamma^b e^{-2\beta_1 l_1}$$

The average power densities can be written as:

$$S_{av}^{i} = \frac{1}{2} Re\left[\bar{E}^i \times \bar{H}^{i*}\right) = \hat{a}_z \frac{|E_o|^2}{2\eta_1}$$

$$S_{av}^{r} = \frac{1}{2} Re\left[\bar{E}^r \times \bar{H}^{r*}\right) = -\hat{a}_z\, |\Gamma^b|^2 \frac{|E_o|^2}{2\eta_1} = -\hat{a}_z\, |\Gamma^b|^2 S_{av}^{i}$$

$$S_{av}^{t} = \frac{1}{2} Re\left(\bar{E}^t \times \bar{H}^{t*}\right) = \hat{a}_z\, |T^b|^2 \frac{|E_o|^2}{2\eta_2} = \hat{a}_z\, |T^b|^2 \frac{\eta_1}{\eta_2} \frac{|E_o|^2}{2\eta_1}$$

$$= \hat{a}_z \left(1 - |\Gamma^b|^2\right) S_{av}^{i}$$

Since the total tangential components of the field intensities on either side must be continuous across the boundary, it is expected that transmitted fields can be greater than the incident field and that will require a transmission coefficient greater than unity. However, by conservation of power, it is well known that the transmitted power density cannot exceed the incident power density

In medium 1 the total Electric and Magnetic field can be written as:

$$\vec{E}^1 = \vec{E}^i + \vec{E}^r = \hat{a}_x \underbrace{E_0 \, e^{-j\beta_1 z}}_{\text{traveling wave}} \underbrace{(1 + \Gamma^b e^{j2\beta_1 z})}_{\text{standing wave}}$$

$$\vec{H}^1 = \vec{H}^i + \vec{H}^r = \hat{a}_y \underbrace{\frac{E_0}{\eta_1} e^{-j\beta_1 z}}_{\text{traveling wave}} \underbrace{(1 - \Gamma^b e^{j2\beta_1 z})}_{\text{standing wave}}$$

Total field of two waves is the product of one of waves times a factor that in this case is the standing wave pattern.

The standing wave ratio (SWR) is defined as the ratio of the maximum value of the electric field magnitude to that of the minimum.

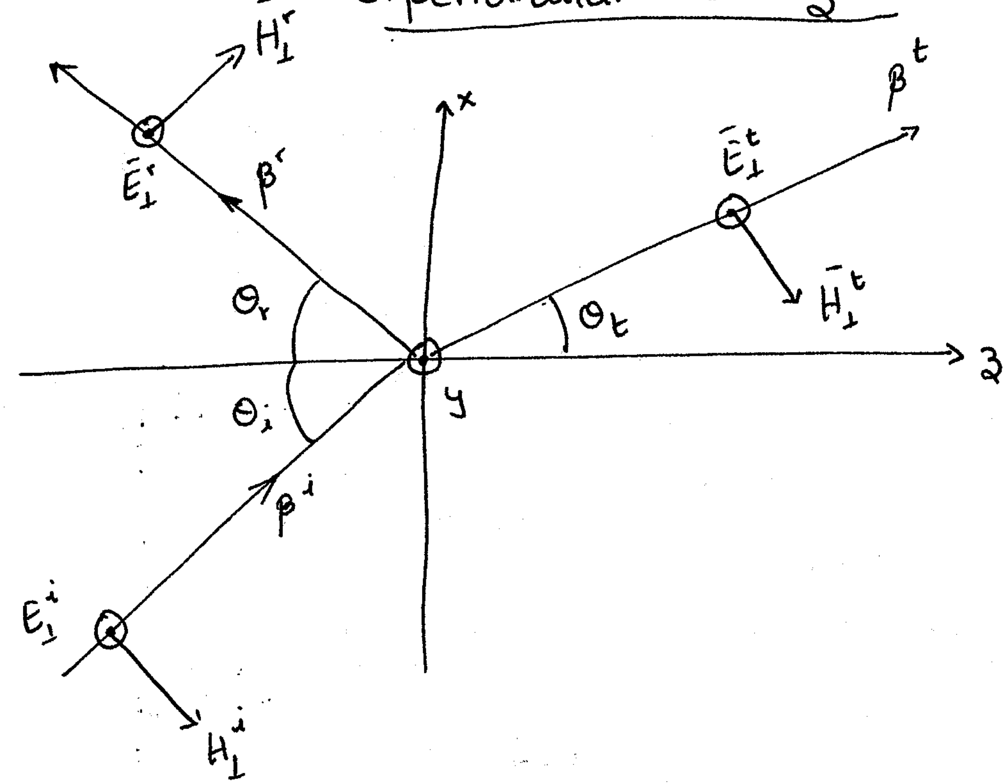
$$SWR = \frac{|\vec{E}^1|_{max}}{|\vec{E}^1|_{min}} = \frac{1 + |\Gamma^b|}{1 - |\Gamma^b|} = \frac{1 + \left|\dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right|}{1 - \left|\dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right|}$$

OBLIQUE INCIDENCE - LOSSLESS MEDIA

Plane of incidence is defined as the plane formed by a unit vector normal to the reflecting interface and the vector in the direction of incidence. To determine the reflections and transmissions at oblique angles of incidence for a general wave polarization, it is most convenient to decompose the electric field into its perpendicular and parallel components and analyze each one of them individually. The total reflected and transmitted field will be the vector sum from each one of these two polarizations.

When the electric field is perpendicular to the plane of incidence, the polarization of the wave is referred to as perpendicular polarization. Since the electric field is parallel to the interface, it is also known as horizontal or E polarization. When the electric field is parallel to the plane of incidence, the polarization is referred to as parallel polarization. Because a component of the electric field is also perpendicular to the interface when the magnetic field is palallel to the interface, it is also known as vertical or H polarization.

Perpendicular Polarization



$$\bar{E}_\perp^i = E_\perp^i \hat{a}_y \, e^{-j\beta_1 (x \sin\theta_i + z \cos\theta_i)}$$

$$H_\perp^i = \frac{E_\perp^i}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$E_\perp^r = \Gamma_\perp E_\perp^i \hat{a}_y \, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$H_\perp^r = \Gamma_\perp \frac{E_\perp^i}{\eta_1}(x\cos\theta_r + z\sin\theta_r)\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$E_\perp^t = T_\perp E_\perp^i \hat{a}_y \, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$H_\perp^t = T_\perp \frac{E_\perp^i}{\eta_2}(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$E_\perp^t = T_\perp^b E_\perp^i = T_\perp^b E_0$$

$$H_\perp^t = \frac{E_\perp^t}{\eta_2} = \frac{T_\perp^b E_0}{\eta_2}$$

Along the surface of the two media by applying boundary conditions the tangential components of electric and magnetic fields are continuous

$$(E_\perp^i + E_\perp^r)\Big|_{\substack{tan \\ z=0}} = (E_\perp^t)\Big|_{\substack{tan \\ z=0}}$$

$$(H_\perp^i + H_\perp^r)\Big|_{\substack{tan \\ z=0}} = (H_\perp^t)\Big|_{\substack{tan \\ z=0}}$$

$$e^{-j\beta_1 x \sin\theta_i} + \Gamma_\perp^b e^{-j\beta_1 x \sin\theta_r} = T_\perp^b e^{-j\beta_2 x \sin\theta_t}$$

$$\frac{1}{\eta_1}(-\cos\theta\, e^{-j\beta_1 x \sin\theta_i} + \Gamma_\perp^b \cos\theta_r\, e^{-j\beta_1 x \sin\theta_r}) = -\frac{T_\perp^b}{\eta_2} \cos\theta_t\, e^{-j\beta_2 x \sin\theta_t}$$

Equating corresponding real and imaginary parts we get two equations

$\theta_i = \theta_r$ (Snell's law for reflection)

$\beta_1 \sin\theta_i = \beta_2 \sin\theta_t$ (Snell's law for refraction)

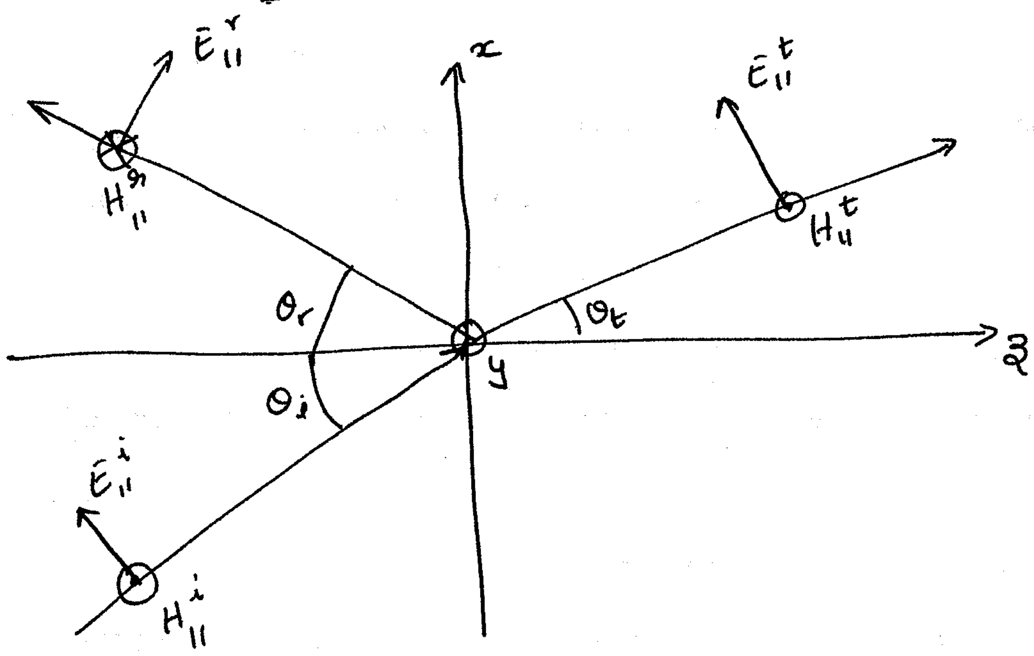
$$1 + \Gamma_\perp^b = T_\perp^b$$

$$\frac{\cos\theta_i}{\eta_1}(-1 + \Gamma_\perp^b) = -\frac{\cos\theta_t}{\eta_2} T_\perp^b$$

$$\Gamma_\perp^b = \frac{E_\perp^r}{E_\perp^i} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_i - \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_t}{\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_i + \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_t}$$

$$T_\perp^b = \frac{E_\perp^t}{E_\perp^i} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{2\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_i}{\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_i + \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_t}$$

Parallel Polarization



Write expression (H/W)

$$E_{\parallel}^{i} = (\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)\, E_o\, e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

$$H_{\parallel}^{i} = \hat{a}_y \frac{E_o}{\eta_1} e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

where $E_{\parallel}^{i} = E_o$ & $H_{\parallel}^{i} = \frac{E_{\parallel}^{i}}{\eta_1} = \frac{E_o}{\eta_1}$

Similarly

$$E_{\parallel}^{t} = (\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t)\, E_{\parallel}^{t}\, e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

$$H_{\parallel}^{t} = \hat{a}_y \frac{T_{\parallel}^{b} E_o}{\eta_2} e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

&

$$E_{\parallel}^{r} = (\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r)\, \Gamma_{\parallel}^{b} E_o e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$

$$H_{\parallel}^{r} = -\hat{a}_y \frac{\Gamma_{\parallel}^{b} E_o}{\eta_1} e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$

$$\Gamma_{\parallel}^{b} = \frac{-\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} = \frac{-\sqrt{\frac{\mu_1}{\varepsilon_1}} \cos\theta_i + \sqrt{\frac{\mu_2}{\varepsilon_2}} \cos\theta_t}{\sqrt{\frac{\mu_1}{\varepsilon_1}} \cos\theta_i + \sqrt{\frac{\mu_2}{\varepsilon_2}} \cos\theta_t}$$

$$T_{\parallel}^{b} = \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} = \frac{2\sqrt{\frac{\mu_2}{\varepsilon_2}} \cos\theta_i}{\sqrt{\frac{\mu_1}{\varepsilon_1}} \cos\theta_i + \sqrt{\frac{\mu_2}{\varepsilon_2}} \cos\theta_t}$$

Total Transmission - Brewster Angle

$$T_\perp^b = \frac{\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_i - \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_t}{\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_i + \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_t} = 0$$

or

$$\cos\theta_i = \sqrt{\frac{\mu_1}{\mu_2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)}\cos\theta_t$$

Using Snell's law of refraction

$$(1-\sin^2\theta_i) = \frac{\mu_1}{\mu_2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)(1-\sin^2\theta_t)$$

$$(1-\sin^2\theta_i) = \frac{\mu_1}{\mu_2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\left(1-\frac{\mu_1}{\mu_2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\sin^2\theta_i\right)$$

$$\boxed{\sin\theta_i = \sqrt{\frac{\frac{\varepsilon_2}{\varepsilon_1}-\frac{\mu_2}{\mu_1}}{\frac{\mu_1}{\mu_2}-\frac{\mu_2}{\mu_1}}}}$$

Since sine function cannot exceed unity

$$\frac{\varepsilon_2}{\varepsilon_1}-\frac{\mu_2}{\mu_1} \le \frac{\mu_1}{\mu_2}-\frac{\mu_2}{\mu_1}$$

$$\frac{\varepsilon_2}{\varepsilon_1} \le \frac{\mu_1}{\mu_2}$$

If $\mu_1 = \mu_2$

$$\sin\theta_i \Big|_{\mu_1 = \mu_2} = \infty$$

Therefore there exists no real angle θ_i under this condition that will reduce the reflection coefficient to zero. Since the permeability for most dielectric material is almost the same and equal to that of free space, then for these materials there exists no real incidence angle that will reduce the reflection coefficient of $\perp$r polarization to zero

Parallel Polarization

$$\Gamma_{\parallel}^{b} = \frac{-\sqrt{\frac{\mu_1}{\varepsilon_1}} \cos\theta_i + \sqrt{\frac{\mu_2}{\varepsilon_2}} \cos\theta_t}{\sqrt{\frac{\mu_1}{\varepsilon_1}} \cos\theta_i + \sqrt{\frac{\mu_2}{\varepsilon_2}} \cos\theta_t} = 0$$

$$\cos\theta_i = \sqrt{\frac{\mu_2}{\mu_1}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)} \cos\theta_t$$

$$\sin\theta_i = \sqrt{\frac{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1}}{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2}}}$$

$$\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1} \leq \frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2}$$

or

$$\frac{\mu_2}{\mu_1} \geq \frac{\varepsilon_1}{\varepsilon_2}$$

If $\mu_1 = \mu_2$

$$\theta_i = \theta_B = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}\right)$$

The incident angle θ_i which reduces the reflection coefficient for parallel polarization to zero, is referred to as Brewster angle θ_B

Total Reflection - Critical Angle

This section will help in finding the incidence angle that allows total reflection of energy at a planar interface.

Perpendicular Polarization

To see the conditions under which the magnitude of the reflection coefficient is equal to unity, we start by saying that:

$$\frac{\left|\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_i - \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_t\right|}{\left|\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_i + \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_t\right|} = 1$$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1 - \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i} = -j\sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i - 1}$$

For this to hold

$$\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i \geq 1$$

<u>or</u>

$$\theta_i \geq \theta_c = \sin^{-1}\left(\sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}\right)$$

The incidence angle θ_i that allows total reflection is known as the critical angle. Since the argument of the sine function cannot exceed unity, then

$$\mu_2\varepsilon_2 \leq \mu_1\varepsilon_1$$

for the critical angle to be physically realizable.

If the permeabilities of the two media are the same.

then

$$\theta_i \geq \theta_c = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

which leads to a physically realizable angle provided.

$$\varepsilon_2 \leq \varepsilon_1$$

Therefore for two media with identical permeabilities, the critical angle exists only if the wave propagates from a more dense to a less dense medium.

"What happens to the angle of refraction and to the propagation of the wave when the angle of incidence is equal to or greater than the critical angle?"

When the angle of incidence is equal to the critical angle, the angle of refraction reduces, Through Snell's law of refraction:

$$\theta_t = \sin^{-1}\left(\sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\sin\theta_i\right)\Bigg|_{\theta_i=\theta_c} = \sin^{-1}\left(\sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}\right) = \sin^{-1}(1) = 90^\circ$$

The reflection & transmission coefficients reduce to

$$\Gamma_\perp^b\Big|_{\theta_i=\theta_c} = 1$$

$$T_\perp^b\Big|_{\theta_i=\theta_c} = 2$$

The transmitted fields can be written as:

$$\bar{E}_\perp^t = \hat{a}_y\, 2E_0\, e^{-j\beta_2 z}$$

$$\bar{H}_\perp^t = \hat{a}_z\, \frac{2E_0}{\eta_2}\, e^{-j\beta_2 z}$$

which represents a plane wave that travels parallel to the interface in the +x direction. This wave is referred to as a surface wave. The average power density associated with the transmitted fields is given by,

$$S_{av}^{t}\Big|_{\theta_i=\theta_c} = \frac{1}{2}\,Re\left(E_{\perp}^{t}\times H_{\perp}^{t*}\right)\Big|_{\theta_i=\theta_c} = \hat{a}_x\,\frac{2|E_0|^2}{\eta_2}$$

and it does not contain any component normal to the interface. Therefore there is no transfer of real power across the interface in a direction normal to the boundary; thus all must be reflected.

$$|S_{av}^{i}|\Big|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1} \qquad |S_{av}^{r}|\Big|_{\theta_r=\theta_c} = \frac{|E_0|^2}{2\eta_1}$$

When angle of incidence is greater than the critical angle ($\theta_i > \theta_c$) then Snell's Law can be written as,

$$\sin\theta_t\Big|_{\theta_i>\theta_c} = \frac{\beta_1}{\beta_2}\sin\theta_i\Big|_{\theta_i>\theta_c} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\,\sin\theta_i\Big|_{\theta_i>\theta_c} > 1$$

which can only be satisfied when θ_t is complex that is

$$\theta_t = \theta_r + j\theta_x$$

&

$$\cos\theta_t\Big|_{\theta_i>\theta_c} = \sqrt{1-\sin^2\theta_t}\Big|_{\theta_i>\theta_c} = \sqrt{1-\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i}\Big|_{\theta_i>\theta_c}$$

$$= \pm j\sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i - 1}\Big|_{\theta_i>\theta_c}$$

When angle of incidence exceeds the critical angle ($\theta_i > \theta_c$) the transmitted $\bar{E}$ field can be written as:

$$E_{\perp}^{t}\Big|_{\theta_i > \theta_c} = \hat{a}_y \, T_{\perp}^{b} \, E_0 \exp(-j\beta_2 x \sin\theta_t) \exp(-j\beta_2 z \cos\theta_t)\Big|_{\theta_i > \theta_c}$$

$$E_{\perp}^{t}\Big|_{\theta_i > \theta_c} = \hat{a}_y \, T_{\perp}^{b} \, E_0 \, e^{-\alpha_e z} \, e^{-j\beta_e x}$$

where

$$\alpha_e = \beta_2 \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2} \sin^2\theta_i - 1}\Bigg|_{\theta_i > \theta_c} = \omega \sqrt{\mu_1 \varepsilon_1 \sin^2\theta_i - \mu_2 \varepsilon_2}\Big|_{\theta_i > \theta_c}$$

$$\beta_e = \beta_2 \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin\theta_i\Bigg|_{\theta_i > \theta_c} = \omega \sqrt{\mu_1 \varepsilon_1} \sin\theta_i\Big|_{\theta_i > \theta_c}$$

$$v_{pe} = \frac{\omega}{\beta_e} = \frac{\omega}{\beta_2 \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin\theta_i}\Bigg|_{\theta_i > \theta_c} = \frac{v_{p2}}{\sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin\theta_i}\Bigg|_{\theta_i > \theta_c}$$

$$= \frac{1}{\sqrt{\mu_1 \varepsilon_1} \sin\theta_i} < v_{p2}$$

The wave associated with above equation also propagates parallel to the interface with constant phase planes that are parallel to the z axis. The effective phase velocity v_{pe} of the wave is less than v_{p_2} of an ordinary wave in medium 2. The ~~effective~~ wave also possesses constant amplitude planes that are $\parallel^{el}$ to the x axis. The effective attenuation constant α_e of the wave in the z direction. Its values are such that the wave decays very rapidly, and in a few wavelengths it essentially vanishes. This wave is also a surface wave. Since its phase velocity is less than the 'speed of light, it is a slow surface wave. Also since it decays very rapidly in a direction normal to the interface, it is tightly bound to the surface or it is a tightly bound slow surface wave.

Chapter-5 (contd)

Lossy Media

For normal incidence in a lossy media we will add an additional term ~~α~~ $e^{-\alpha_1 z}$ & $e^{-\alpha_2 z}$ to the incident a reflected and transmitted electric & magnetic fields in medium 1 & 2 respectively.

for eg:- $\bar{E}^i = \hat{a}_x E_0 e^{-\alpha_1 z} e^{-j\beta_1 z}$

$$\bar{H}^i = \hat{a}_y \frac{E_0}{\eta_1} e^{-\alpha_1 z} e^{-j\beta_1 z}$$

The ~~con~~ attenuation constant α_i, phase constant β_i and intrinsic impedance η_i are related to ε_i, μ_i & σ_i by the following relationship.

	Exact	Good dielectric	Good conductor
α	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]\right\}^{1/2}$	$\approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
β	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]\right\}^{1/2}$	$\simeq \omega\sqrt{\mu\varepsilon}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
η	$= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$	$\simeq \sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$

Total Electric and magnetic fields in medium 1

$$\bar{E}^1 = \bar{E}^i + \bar{E}^r = \hat{a}_x E_o e^{-\alpha_1 z} e^{-j\beta_1 z}\left(1 + \Gamma^b e^{2\alpha_1 z} e^{j2\beta_1 z}\right)$$

$$\bar{H}^1 = \bar{H}^i + \bar{H}^r = \hat{a}_y \frac{E_o}{\eta_1} e^{-\alpha_1 z} e^{-j\beta_1 z}\left(1 - \Gamma^b e^{2\alpha_1 z} e^{j2\beta_1 z}\right)$$

OBLIQUE INCIDENCE: Dielectric Conductor Interface

Assume a uniform plane wave is obliquely incident upon a planar interface where medium 1 is a perfect dielectric and medium 2 is lossy. The transmitted wave can be written as:

$$\bar{E}_\perp^t = \hat{a}_y E_\perp^t e^{-j\beta^t \cdot r} = \hat{a}_y T_\perp^b E_o e^{-j\beta_2(x \sin\theta_t + z \cos\theta_t)}$$

or

$$\bar{E}_\parallel^t = (\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t)\, T_\parallel^b E_o e^{-j\beta_2(x \sin\theta_t + z\cos\theta_t)}$$

In general for lossy medium we can write

$$\bar{E}^t = \bar{E}_2 \exp\left[-\gamma_2 (x \sin\theta_t + z\cos\theta_t)\right] \quad - \quad (1)$$

For lossy media Snell's Law can be written as:

$$\gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t$$

$$\sin\theta_t = \frac{\gamma_1}{\gamma_2} \sin\theta_i = \frac{j\beta_1}{\alpha_2 + j\beta_2} \sin\theta_i$$

& $\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\left(\frac{j\beta_1}{\alpha_2+j\beta_2}\right)^2 \sin^2\theta_i} = s e^{j\xi}$

$= s\,(\cos\xi + j\sin\xi)$

Substituting in (1)

$$\bar{E}^t = \bar{E}_2 \exp\left\{-(\alpha_2+j\beta_2)\left[x\,\frac{j\beta_1}{\alpha_2+j\beta_2}\sin\theta_i + z\,s\,(\cos\xi + j\sin\xi)\right]\right\}$$

which reduces to

$$\bar{E}^t = \bar{E}_2 \exp\left[-zs\,(\alpha_2\cos\xi - \beta_2\sin\xi)\right] \times \exp\left\{-j\left[\beta_1 x\sin\theta_i + zs\,(\alpha_2\sin\xi + \beta_2\cos\xi)\right]\right\}$$

$$\bar{E}^t = \bar{E}_2\, e^{-zp} \exp\left[-j(\beta_1 x\sin\theta_i + zq)\right] \quad \text{— (2)}$$

where

$$\left.\begin{aligned} p &= s\,(\alpha_2\cos\xi - \beta_2\sin\xi) = \alpha_{2e} \\ q &= s\,(\alpha_2\sin\xi + \beta_2\cos\xi) \end{aligned}\right\} \quad \text{— (3)}$$

The instantaneous electric field can be written as:-

$$\bar{\mathcal{E}}^t = Re\left(\bar{E}^t e^{j\omega t}\right) = \bar{E}_2\, e^{-zp}\, Re\left(\exp\left\{j\left[\omega t - (\beta_1 x\sin\theta_i + zq)\right]\right\}\right)$$

$$\bar{\mathcal{E}}^t = \bar{E}_2\, e^{-zp} \cos\left[\omega t - (\beta_1 x\sin\theta_i + zq)\right] \quad \text{— (4)}$$

The constant amplitude planes are parallel to the interface and the constant phase planes are inclined at an angle ψ_2 that is no longer θ_t

We can write

$$\omega t-(\beta_1 x\sin\theta_i+zq)=\omega t-\sqrt{(\beta_1\sin\theta_i)^2+q^2}$$
$$\times\left[\frac{(\beta_1\sin\theta_i)\,x}{\sqrt{(\beta_1\sin\theta_i)^2+q^2}}+\frac{q z}{\sqrt{(\beta_1\sin\theta_i)^2+q^2}}\right] \quad \text{— (5)}$$

If we define an angle ψ_2 such that

$$u=\beta_1\sin\theta_i$$

$$\sin\psi_2=\frac{\beta_1\sin\theta_i}{\sqrt{(\beta_1\sin\theta_i)^2+q^2}}=\frac{u}{\sqrt{u^2+q^2}} \quad \text{— (6)}$$

$$\cos\psi_2=\frac{q}{\sqrt{(\beta_1\sin\theta_i)^2+q^2}}=\frac{q}{\sqrt{u^2+q^2}} \quad \text{— (7)}$$

or we can write

$$\psi_2=\tan^{-1}\left(\frac{\beta_1\sin\theta_i}{q}\right)=\tan^{-1}\left(\frac{u}{q}\right) \quad \text{— (8)}$$

We can write (2) as:

$$\bar{\mathcal{E}}^t=\bar{E}_z e^{-zp}\,Re\left[\exp\left\{j\left[\omega t-\sqrt{u^2+q^2}\left(\frac{ux}{\sqrt{u^2+q^2}}+\frac{qz}{\sqrt{u^2+q^2}}\right)\right]\right\}\right)$$

$$=\bar{E}_z e^{-zp}\,Re\left[\exp\left\{j\left[\omega t-\beta_{2e}(x\sin\psi_2+z\cos\psi_2)\right]\right\}\right]$$

$$\bar{\mathcal{E}}^t=\bar{E}_z e^{-zp}\,Re\left[\exp\left\{j\left[\omega t-\beta_{2e}(\hat{n}_\psi\cdot\bar{r})\right]\right\}\right) \quad \text{— (9)}$$

where

$$\left.\begin{aligned} \hat{n}_\psi &= \hat{a}_x \sin\psi_2 + \hat{a}_z \cos\psi_2 \\ \beta_{2e} &= \sqrt{u^2+q^2} \end{aligned}\right\} \quad - \quad (10)$$

The true angle of refraction is ψ_2 and not θ_2

The wave travels along a direction defined by unit vector $\hat{n}_\psi$

The constant phase planes are $\perp^r$ to unit vector $\hat{n}_\psi$.

The phase velocity of the wave in medium 2 is obtained by setting the exponent of (9) to a constant and differentiating it with respect to time. We get.

$$\omega(1) - \sqrt{u^2+q^2}\left(\hat{n}_\psi \cdot \frac{d\bar{r}}{dt}\right) = 0$$

$$\omega(1) - \sqrt{u^2+q^2}\left(\hat{n}_\psi \cdot \frac{d\bar{r}}{dt}\right) = \omega - \beta_{2e}(\hat{n}_\psi \cdot v_p) = 0$$

or

$$v_{pr} = \frac{\omega}{\beta_{2e}} = \frac{\omega}{\sqrt{u^2+q^2}} = \frac{\omega}{(\beta_1 \sin\theta_i)^2 + q^2}$$

OBLIQUE INCIDENCE : CONDUCTOR - CONDUCTOR INTERFACE

Read T.B pg 214 – 220