Solution of Wave Equation (contd)

Source free and Lossy Media

For a lossy media ($\sigma \neq 0$) but source free ($\bar{J}_i = \bar{M}_i = q_{ve} = q_{vm} = 0$), the vector wave equations that $\bar{E}$ and $\bar{H}$ must satisfy is given by:-

$$\nabla^2 \bar{E} = j\omega\mu\sigma \bar{E} - \omega^2 \mu\varepsilon \bar{E} = \gamma^2 \bar{E} \qquad -(1)$$

$$\nabla^2 \bar{H} = j\omega\mu\sigma \bar{H} - \omega^2 \mu\varepsilon \bar{H} = \gamma^2 \bar{H}$$

The general solution can be given as

$$\bar{E}(x,y,z) = \hat{a}_x E_x(x,y,z) + \hat{a}_y E_y(x,y,z) + \hat{a}_z E_z(x,y,z) \qquad -(2)$$

Substituting (2) into (1) we get

$$\nabla^2 \bar{E} - \gamma^2 \bar{E} = 0$$

$$\nabla^2(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) - \gamma^2(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) = 0 \qquad -(3)$$

which reduces to three scalar wave equations of

$$\left.\begin{aligned} \nabla^2 E_x(x,y,z) - \gamma^2 E_x(x,y,z) &= 0 \\ \nabla^2 E_y(x,y,z) - \gamma^2 E_y(x,y,z) &= 0 \\ \nabla^2 E_z(x,y,z) - \gamma^2 E_z(x,y,z) &= 0 \end{aligned}\right\} \qquad -(4)$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$\gamma = \pm\sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \begin{cases} \pm(\alpha + j\beta) & \text{for } +\sigma \\ \pm(\alpha - j\beta) & \text{for } -\sigma \end{cases}$$

There are four possible combinations for the form of 'γ'

$$\gamma = \begin{cases} +(\alpha + j\beta) \\ -(\alpha + j\beta) \\ +(\alpha - j\beta) \\ -(\alpha - j\beta) \end{cases} \qquad -(5)$$

We can write the solution ~~as~~ using separation of variables as:-

$$E_x(x, y, z) = f(x)\, g(y)\, h(z) \qquad -(6)$$

The solution of these variables can be written as:

$$\left.\begin{aligned} f_1(x) &= A_1 e^{-\gamma_x x} + B_1 e^{\gamma_x x} \\ &\underline{\underline{\text{OR}}} \\ f_2(x) &= C_1 \cosh(\gamma_x x) + D_1 \sinh(\gamma_x x) \end{aligned}\right\} \qquad (7)$$

& $g(y)$ can be written as

$$\left.\begin{aligned} g_1(y) &= A_2 e^{-\gamma_y y} + B_2 e^{\gamma_y y} \\ &\underline{\underline{\text{OR}}} \\ g_2(y) &= C_2 \cosh(\gamma_y y) + D_2 \sinh(\gamma_y y) \end{aligned}\right\} \qquad (8)$$

& $h(z)$ can be written as

$$\left.\begin{aligned} h_1(z) &= A_3 e^{-\gamma_z z} + B_3 e^{\gamma_z z} \\ &\underline{\underline{\text{OR}}} \\ h_2(z) &= C_3 \cosh(\gamma_z z) + D_3 \sinh(\gamma_z z) \end{aligned}\right\} \qquad (9)$$

The constraint equation can be written as:-

$$\gamma_x^2+\gamma_y^2+\gamma_z^2=\gamma^2 \qquad -\ (10)$$

exponentials represent attenuating traveling waves & hyperbolic cosines and sines represent attenuating standing waves.

If we assume the wave propagation in 'z' direction:-

$$\gamma_z = \begin{cases} +\ (\alpha_z+j\beta_z) \\ -\ (\alpha_z+j\beta_z) \\ +\ (\alpha_z-j\beta_z) \\ -\ (\alpha_z-j\beta_z) \end{cases} \qquad -\ (10)$$

Taking equation (9) we have

$$h_1^+(z) = \left\{\begin{matrix} A_z e^{-\gamma_z z} = A_z e^{-\alpha_z z} e^{-j\beta_z z} \\ A_z e^{-\gamma_z z} = A_z e^{\alpha_z z} e^{j\beta_z z} \\ A_z e^{-\gamma_z z} = A_z e^{-\alpha_z z} e^{j\beta_z z} \\ A_z e^{-\gamma_z z} = A_z e^{\alpha_z z} e^{-j\beta_z z} \end{matrix}\right\} \qquad -\ (11)$$

increasing or decreasing amplitudes

direction of propagation

Cylindrical Coordinate System

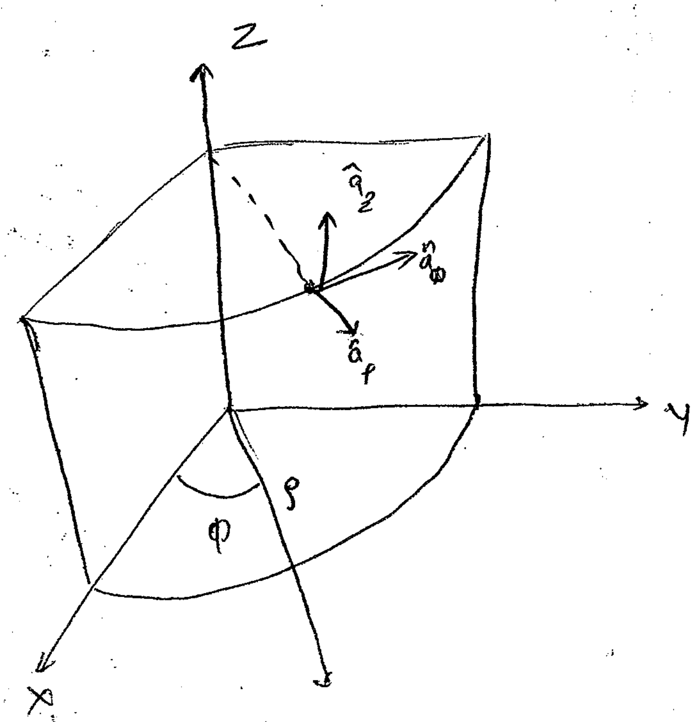
The general solution to vector wave equation for source free & lossless media ~~the~~ can be written as:

$$\bar{E}(\rho, \phi, z) = \hat{a}_\rho E_\rho(\rho, \phi, z) + \hat{a}_\phi E_\phi(\rho, \phi, z) + \hat{a}_z E_z(\rho, \phi, z)$$

ρ, ϕ & z are cylindrical coordinates

$$\nabla^2 \bar{E} = -\beta^2 \bar{E}$$

$$\nabla^2 (\hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z) = -\beta^2 (\hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z$$



Note that like in Rectangular coordinate system these donot reduce to three simple scalar wave equations because we cannot write

$$\nabla^2 (\hat{a}_\rho E_\rho) \neq \hat{a}_\rho \nabla^2 E_\rho \quad \&$$

$$\nabla^2 (\hat{a}_\phi E_\phi) \neq \hat{a}_\phi \nabla^2 E_\phi$$

If we assume two points (ρ_1, ϕ_1, z_1) & (ρ_2, ϕ_2, z_2) and their corresponding vectors in cylindrical coordinate system we observe that the direction of $\hat{a}_\rho$ and $\hat{a}_\phi$ have changed from one point to another and therefore cannot be treated as constants but are functions of ρ, ϕ & z.

In the z direction the equation can be written as

$$\nabla^2 E_z + \beta^2 E_z = 0 \quad \rightarrow \quad \nabla^2 E_z = -\beta^2 E_z$$

$\nabla^2 \bar{E}$ can be written as

$$\nabla^2 \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla \times \nabla \times \bar{E}$$

$$\nabla(\nabla \cdot \bar{E}) - \nabla \times \nabla \times \bar{E} = -\beta^2 \bar{E} \qquad \text{— (A)}$$

Substituting

$$\bar{E}(\rho, \phi, z) = \hat{a}_\rho E_\rho(\rho, \phi, z) + \hat{a}_\phi E_\phi(\rho, \phi, z) + \hat{a}_z E_z(\rho, \phi, z)$$

into equation (A)

$$\nabla^2 E_\rho + \left(-\frac{E_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial E_\phi}{\partial \phi} \right] = -\beta^2 E_\rho \qquad \text{— (1)}$$

$$\nabla^2 E_\phi + \left[-\frac{E_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \phi} \right) = -\beta^2 E_\phi \qquad \text{— (2)}$$

$$\nabla^2 E_z = -\beta^2 E_z \qquad \text{— (3)}$$

$$\nabla^2 \psi(\rho, \phi, z) = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \qquad \text{— (4)}$$

$$= \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Equation (1) & (2) are coupled second-order partial differential equations. Eq 3 is an uncoupled second-order partial differential epuation which is used for generating TE^z & TM^z mode solutions of boundary-value problems

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = -\beta^2\psi \qquad -(3)$$

$\psi(\rho,\phi,z)$ is a scalar function that can represent a field or a vector potential component. Assuming a separable solution for $\psi(\rho,\phi,z)$ of the form

$$\psi(\rho,\phi,z) = f(\rho)\,g(\phi)\,h(z) \qquad -(4)$$

& substituting (4) in (3)

$$gh\frac{\partial^2 f}{\partial\rho^2} + gh\frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{fh}{\rho^2}\frac{\partial^2 g}{\partial\phi^2} + fg\frac{\partial^2 h}{\partial z^2} = -\beta^2 fgh \qquad -(5)$$

Dividing (5) by fgh

$$\frac{1}{f}\frac{\partial^2 f}{\partial\rho^2} + \frac{1}{f}\frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{g}\frac{1}{\rho^2}\frac{\partial^2 g}{\partial\phi^2} + \frac{1}{h}\frac{\partial^2 h}{\partial z^2} = -\beta^2 \qquad -(6)$$

Using last term of (6), we have

$$\frac{1}{h}\frac{\partial^2 h}{\partial z^2} = -\beta_z^2 \implies \frac{\partial^2 h}{\partial z^2} = -h\beta_z^2 \qquad -(7)$$

Substituting (7) into (6)

$$\frac{1}{f}\frac{\partial^2 f}{\partial\rho^2} + \frac{1}{f}\frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{g}\frac{1}{\rho^2}\frac{\partial^2 g}{\partial\phi^2} \cancel{+\frac{1}{h}\frac{\partial^2 h}{\partial z^2}} - \beta_z^2 + \beta^2 = 0 \qquad -(8)$$

Multiplying (8) by ρ^2

$$\frac{\rho^2}{f}\frac{\partial^2 f}{\partial \rho^2}+\frac{\rho}{f}\frac{\partial f}{\partial \rho}+\frac{1}{g}\frac{\partial^2 g}{\partial \phi^2}+(\beta^2-\beta_z^2)\rho^2=0 \qquad -(9)$$

Setting $\frac{1}{g}\frac{\partial^2 g}{\partial \phi^2}$ to a constant $-m^2$

$$\frac{1}{g}\frac{\partial^2 g}{\partial \phi^2}=-m^2 \Longrightarrow \frac{\partial^2 g}{\partial \phi^2}=-m^2 g \qquad -(10)$$

Letting

$$\beta^2-\beta_z^2=\beta_\rho^2 \Rightarrow \beta_\rho^2+\beta_z^2=\beta^2 \qquad -(11)$$

Substituting (11) to (9) & multiplying by f

$$\rho^2\frac{\partial^2 f}{\partial \rho^2}+\rho\frac{\partial f}{\partial \rho}+\left[(\beta_\rho\rho)^2-m^2\right]f=0 \qquad -(12)$$

Equation (11) is referred to as constraint equation for solution to the wave equation in cylindrical coordinates equation (12) is the classic Bessel differential equation

$$\psi(\rho,\phi,z)=f(\rho)\,g(\phi)\,h(z)$$

reduces to

$$\left.\begin{aligned}
&\rho^2\frac{\partial^2 f}{\partial \rho^2}+\rho\frac{\partial f}{\partial \rho}+\left[(\beta_\rho\rho)^2-m^2\right]f=0\\
&\frac{\partial^2 g}{\partial \phi^2}=-m^2 g\\
&\frac{\partial^2 h}{\partial z^2}=-\beta_z^2 h
\end{aligned}\right\} \qquad -(13)$$

& $\beta_\rho^2 + \beta_z^2 = \beta^2$ — (14)

Solution to eqn (13) ~~also~~ takes the form

$$f_1(\rho) = A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)$$

OR

$$f_2(\rho) = C_1 H_m^{(1)}(\beta_\rho \rho) + D_1 H_m^{(2)}(\beta_\rho \rho)$$

and

$$g_1(\phi) = A_2 e^{-jm\phi} + B_2 e^{jm\phi}$$

or

$$g_2(\phi) = C_2 \cos(m\phi) + D_2 \sin(m\phi)$$

and

$$h_1(z) = A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}$$

or

$$h_2(z) = C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)$$

$J_m(\beta_\rho \rho)$ and $Y_m(\beta_\rho \rho)$ represent the Bessel function of the first and second kind respectively; $H_m^{(1)}(\beta_\rho \rho)$ and $H_m^{(2)}(\beta_\rho \rho)$ represents the Hankel functions of the first and second kind. To represent the fields in the region outside the cylindar, a typical solution for $\psi(\rho, \phi, z)$ would take the form

$$\psi_2(\rho, \phi, z) = B_1 H_m^{(2)}(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right]\left[A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}\right]$$