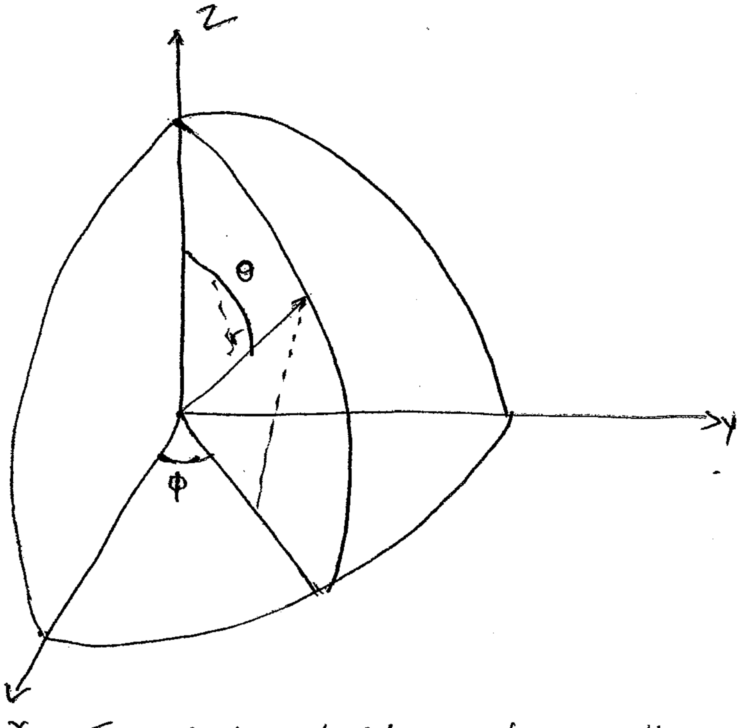
Spherical Coordinate System



The electric field can be written as

$$\bar{E}(r,\theta,\phi) = \hat{a}_r E_r(r,\theta,\phi) + \hat{a}_\theta E_\theta(r,\theta,\phi) + \hat{a}_\phi E_\phi(r,\theta,\phi) \quad -(1)$$

For a lossless medium

$$\nabla^2 \bar{E} = -\beta^2 \bar{E}$$

$$\nabla^2(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) = -\beta^2(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi)$$

These terms will not reduce to 3 scalar equations.

Using the relationship (identity)

$$\nabla \times \nabla \times \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} \quad -(2)$$

$$\nabla^2 \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla \times \nabla \times \bar{E} \quad -(3)$$

Substituting (3) into (1) we get

$$\nabla^2 E_r - \frac{2}{r^2}\left(E_r + E_\theta \cot\theta + \operatorname{cosec}\theta \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_\theta}{\partial \theta}\right) = -\beta^2 E_r \quad -(4)$$

$$\nabla^2 E_\theta - \frac{1}{r^2}\left[E_\theta \operatorname{cosec}^2\theta - 2\frac{\partial E_r}{\partial \theta} + 2\cot\theta \operatorname{cosec}\theta \frac{\partial E_\phi}{\partial \phi}\right] = -\beta^2 E_\theta \quad -(5)$$

$$\nabla^2 E_\phi - \frac{1}{r^2}\left[E_\phi \operatorname{cosec}^2\theta - 2\operatorname{cosec}\theta \frac{\partial E_r}{\partial \phi} - 2\cot\theta \operatorname{cosec}\theta \frac{\partial E_\theta}{\partial \phi}\right] = -\beta^2 E_\phi \quad -(6)$$

All the three epuations (4) - (6) are coupled partial differential equations.

The transverse electric TE^r and magnetic TM^r wave mode solutions can be formed that in spherical coordinates must satisfy the scalar wave epuation of.

$$\nabla^2 \psi(r,\theta,\phi) = -\beta^2 \psi(r,\theta,\phi) \quad -(7)$$

$\psi(r,\theta,\phi) \rightarrow$ scalar function that can represent a field or a vector potential component.

Assuming a separable solution for $\psi(r,\theta,\phi)$ of the form

$$\psi(r,\theta,\phi) = f(r)\, g(\theta)\, h(\phi) \quad -(8)$$

we can write the expansion of (7) as.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left\{\sin\theta \frac{\partial \psi}{\partial \theta}\right\} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = -\beta^2 \psi \quad -(9)$$

Substituting (8) into (9)

$$gh\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + fh\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left\{\sin\theta\frac{\partial g}{\partial \theta}\right\} + fg\frac{1}{r^2\sin^2\theta}\frac{\partial^2 h}{\partial \phi^2} = -\beta^2 fgh \quad -(10)$$

$\div$ by fgh and multiplying by $r^2\sin^2\theta$

$$\frac{\sin^2\theta}{f}\frac{d}{dr}\left\{r^2\frac{df}{dr}\right\} + \frac{\sin\theta}{g}\frac{d}{d\theta}\left\{\sin\theta\frac{dg}{d\theta}\right\} + \frac{1}{h}\frac{d^2 h}{d\phi^2} = -(\beta r\sin\theta)^2 \quad -(11)$$

Since the last term is only a function of 'ϕ' it can be set equal to

$$\frac{1}{h}\frac{d^2h}{d\phi^2} = -m^2 \Rightarrow \frac{d^2h}{d\phi^2} = -m^2 h \qquad -(12)$$

Substituting (12) into (11) we get

$$\frac{1}{f}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + (\beta r)^2 + \frac{1}{g\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) - \left(\frac{m}{\sin\theta}\right)^2 = 0 \qquad (13)$$

Since last two terms are only a function of θ, we can set them equal to

$$\frac{1}{g\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) - \left(\frac{m}{\sin\theta}\right)^2 = -n(n+1) \qquad -(14)$$

Substituting (14) into (13)

$$\frac{1}{f}\frac{d}{dr}\left\{r^2\frac{df}{dr}\right\} + (\beta r)^2 - (n(n+1)) = 0 \qquad -(15)$$

which is closely related to Bessel differential equation

Summary

The scalar wave equation in spherical coordinates can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial\psi}{\partial r}\right\} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial\psi}{\partial\theta}\right\} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = -\beta^2\psi \qquad -(16)$$

and the separable solution takes the form

$$\psi(r,\theta,\phi) = f(r)\,g(\theta)\,h(\phi)$$

reduces to three scalar differential equations.

$$\frac{d}{dr}\left\{r^2\frac{df}{dr}\right\}+\left[(\beta r)^2-n(n+1)\right]f=0 \qquad -(17)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{dg}{d\theta}\right\}+\left[n(n+1)-\left\{\frac{m}{\sin\theta}\right\}^2\right]g=0 \qquad -(18)$$

$$\frac{d^2h}{d\phi^2}=-m^2h \qquad -(19)$$

Solution of these equations can be written as

$$f_1(r)=A_1 j_n(\beta r)+B_1 y_n(\beta r)$$

or

$$f_2(r)=C_1 h_n^{(1)}(\beta r)+D_1 h_n^{(2)}(\beta r)$$

and

$$g_1(\theta)=A_2 P_n^m(\cos\theta)+B_2 P_n^m(-\cos\theta) \qquad n\neq\text{integer}$$

or

$$g_2(\theta)=C_2 P_n^m(\cos\theta)+D_2 Q_n^m(\cos\theta) \qquad n=\text{integer}$$

and

$$h_1(\phi)=A_3 e^{-jm\phi}+B_3 e^{jm\phi}$$

or

$$h_2(\phi)=C_3\cos(m\phi)+D_3\sin(m\phi)$$

$j_n(\beta r)$ and $y_n(\beta r)$ are spherical Bessel functions of the first and second kind

They are used for representing radial standing waves, and are related to the regular Bessel functions $J_{n+1/2}(\beta r)$ and $Y_{n+1/2}(\beta r)$ by

$$j_n(\beta r)=\sqrt{\frac{\pi}{2\beta r}}\,J_{n+1/2}(\beta r)$$

Ro

$$y_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, Y_{n+1/2}(\beta r)$$

$h_n^{(1)}(\beta r)$ and $h_n^{(2)}(\beta r)$ are referred to as spherical Hankel functions of the first and second kind. They are used to represent traveling waves and related to the regular Hankel functions as:

$$h_n^{(1)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, H^{(1)}_{n+1/2}(\beta r)$$
$$h_n^{(2)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, H^{(2)}_{n+1/2}(\beta r)$$

$P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$ are the assocoated Legendre functions of the first and second kind.

A typical solution for $\psi(r,\theta,\phi)$ to represent the fields within a sphere may take the form

$$\psi_1(r,\theta,\phi) = [A_1 j_n(\beta r) + B_1 y_n(\beta r)][C_2 P_n^m(\cos\theta) + D_2 Q_n^m(\cos\theta)][C_3\cos(m\phi) + D_3\sin(m\phi)] \quad -(19)$$

For fields to be finite at $r=0$, where $y_n(\beta r)$ possesses a singularity, and for any value of θ, including $\theta=0,\pi$ where $Q_n^m(\cos\theta)$ possesses singularities (19) reduces to

$$\psi_1(r,\theta,\phi) = A_{mn} j_n(\beta r)\, P_n^m(\cos\theta)[C_3\cos(m\phi) + D_3\sin(m\phi)] \quad -(20)$$

For the fields outside the sphere the solution takes the form

$$\psi_2(r,\theta,\phi) = B_{mn}\, h_n^{(2)}(\beta r)\, P_n^m(\cos\theta)\,[C_3\cos(m\phi) + D_3\sin(m\phi)] \quad -(21)$$

The spherical Hankel function of the second kind $h_n^{(2)}(\beta r)$ has replaced the spherical Bessel function of the first kind $j_n(\beta r)$ because outward traveling waves are formed outside the sphere, in contrast to the standing waves inside the sphere.

Polarization

It is defined as that property of a radiated electromagnetic wave describing the time-varying direction and relative magnitude of the electric field vector; specifically, the figure traced as a function of time by the extremity of the vector at a fixed location in space, and the sense in which it is traced, as observed along the direction of propagation. Polarization is the curve traced out by the end point of the arrow representing the instantaneous electric field.

Polarization can be classified into linear, circular and elliptical polarization. If the vector that describes the electric field at a point in space as a function of time is always directed along a line, which is normal to the direction of propagation, the field is said to be linearly polarized. The figure that the electric field traces is an ellipse, and the field is said to be elliptically polarized. Linear & circular polarizations are special cases of elliptical polarization. The figure of electric field is traced in a clockwise or counterclockwise sense. Clockwise rotation of the electric field is designated as right-hand polarization and counterclockwise as left hand polarization

Linear Polarization

Assume a harmonic plane wave, with x and y electric field components, traveling in the +ve z direction

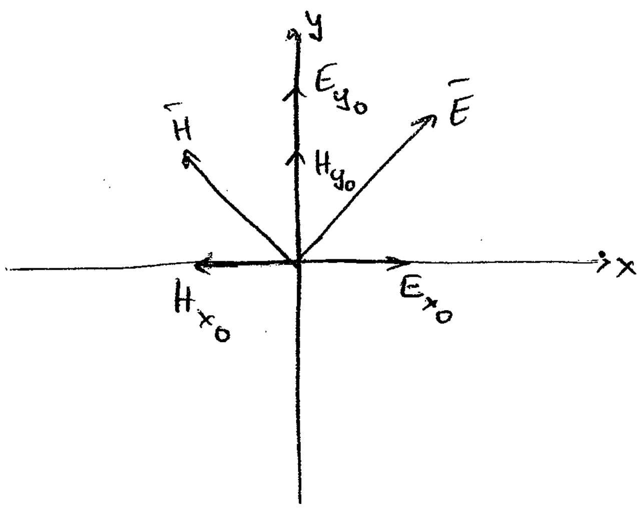
The instantaneous electric and magnetic fields are given by

$$\mathcal{E} = \hat{a}_x \mathcal{E}_x + \hat{a}_y \mathcal{E}_y = Re\left[\hat{a}_x E_x^+ e^{j(\omega t - \beta z)} + \hat{a}_y E_y^+ e^{j(\omega t - \beta z)}\right]$$

$$= \hat{a}_x E_{x_0}^+ \cos(\omega t - \beta z + \phi_x) + \hat{a}_y E_{y_0}^+ \cos(\omega t - \beta z + \phi_y) \quad -(1)$$

$$\mathcal{H} = \hat{a}_y \mathcal{H}_y + \hat{a}_x \mathcal{H}_x = Re\left[\hat{a}_y \frac{E_x^+}{\eta} e^{j(\omega t - \beta z)} - \hat{a}_x \frac{E_y^+}{\eta} e^{j(\omega t - \beta z)}\right]$$

$$= \hat{a}_y \frac{E_x^+}{\eta} \cos(\omega t - \beta z + \phi_x) - \hat{a}_x \frac{E_y^+}{\eta} \cos(\omega t - \beta z + \phi_y) \quad -(2)$$



Linear Polarization

Let us consider a harmonic plane wave, with x and y electric field components, traveling in the +ve z direction. The instantaneous electric and magnetic fields are given as:

$$\mathcal{E} = \hat{a}_x \mathcal{E}_x + \hat{a}_y \mathcal{E}_y = Re\left[\hat{a}_x E_x^+ e^{j(\omega t - \beta z)} + \hat{a}_y E_y^+ e^{j(\omega t - \beta z)}\right]$$

$$= \hat{a}_x E_{x_0}^+ \cos(\omega t - \beta z + \phi_x) + \hat{a}_y E_{y_0}^+ \cos(\omega t - \beta z + \phi_y)$$

$$\mathcal{H} = \hat{a}_y H_y + \hat{a}_x H_x = Re\left[\hat{a}_y \frac{E_x^+}{\eta} e^{j(\omega t - \beta z)} - \hat{a}_x \frac{E_y^+}{\eta} e^{j(\omega t - \beta z)}\right]$$

$$= \hat{a}_y \frac{E_{x_0}^+}{\eta} \cos(\omega t - \beta z + \phi_x) - \hat{a}_x \frac{E_{y_0}^+}{\eta} \cos(\omega t - \beta z + \phi_y)$$

E_x^+ and E_y^+ are complex and $E_{x_0}^+$ and $E_{y_0}^+$ are real.

At $z = 0$ plane. Assume $E_{y_0}^+ = 0$

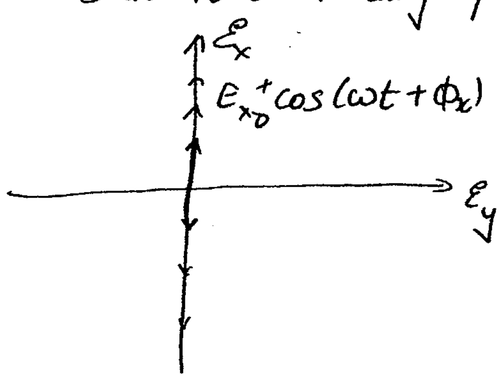
$$\mathcal{E}_x = E_{x_0}^+ \cos(\omega t + \phi_x)$$

$$\mathcal{E}_y = 0$$

The locus of instantaneous electric field vector is given by

$$\mathcal{E} = \hat{a}_x E_{x_0}^+ \cos(\omega t + \phi_x)$$

The field is said to be linearly polarized in the x direction



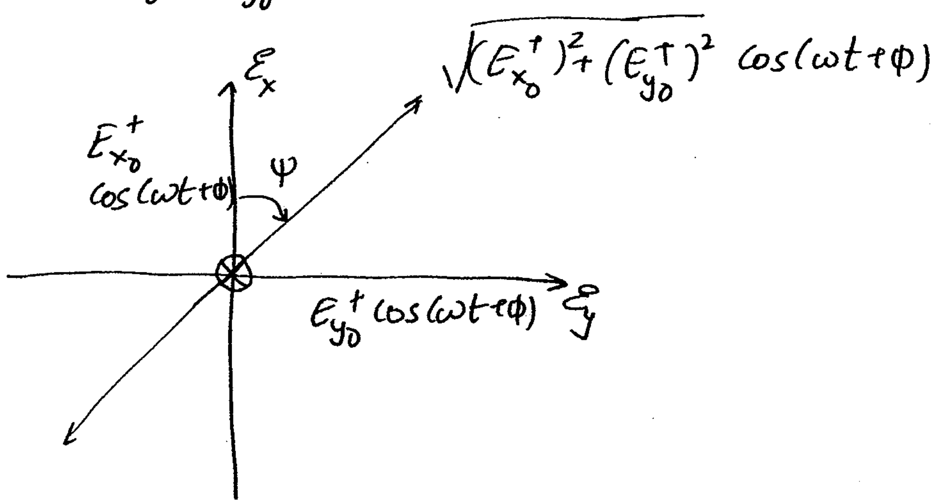
Assume $\phi_x = \phi_y = \phi$

then

$$\mathcal{E}_x = E_{x_0}^{+} \cos(\omega t + \phi) \qquad \mathcal{E}_y = E_{y_0}^{+} \cos(\omega t + \phi)$$

The amplitude of the electric field vector is given by

$$\mathcal{E} = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2} = \left(\sqrt{(E_{x_0}^{+})^2 + (E_{y_0}^{+})^2}\right) \cos(\omega t + \phi)$$



$$\psi = \tan^{-1}\left[\frac{\mathcal{E}_y}{\mathcal{E}_x}\right] = \tan^{-1}\left[\frac{E_{y_0}^{+}}{E_{x_0}^{+}}\right]$$

The field is said to be linearly polarized in the ψ direction.

A time harmonic field is linearly polarized at a given point in space if the electric field (or magnetic field) vector at that point is always oriented along the same straight line at every instant of time.

Circular Polarization

A wave is said to be circularly polarized if the tip of the electric field vector traces out a circular locus in space

Right hand Circular Polarization

A wave has right hand circular polarization if its electric field vector has a clockwise sense of rotation when it is viewed along the axis of propagation. The instantaneous electric field can be written as.

$$\mathcal{E} = \hat{a}_x \mathcal{E}_x + \hat{a}_y \mathcal{E}_y = Re\left[\hat{a}_x E_x^+ e^{j(\omega t - \beta z)} + \hat{a}_y E_y^+ e^{j(\omega t - \beta z)}\right]$$

$$= \hat{a}_x E_{x_0}^+ \cos(\omega t - \beta z + \phi_x) + \hat{a}_y E_{y_0}^+ \cos(\omega t - \beta z + \phi_y)$$

At $z=0$ plane. Assume $\phi_x = 0$ and $\phi_y = -\frac{\pi}{2}$

$$E_{x_0}^+ = E_{y_0}^+ = E_R$$

Then

$$\mathcal{E}_x = E_R \cos(\omega t) \qquad \mathcal{E}_y = E_R \cos(\omega t - \tfrac{\pi}{2}) = E_R \sin(\omega t)$$

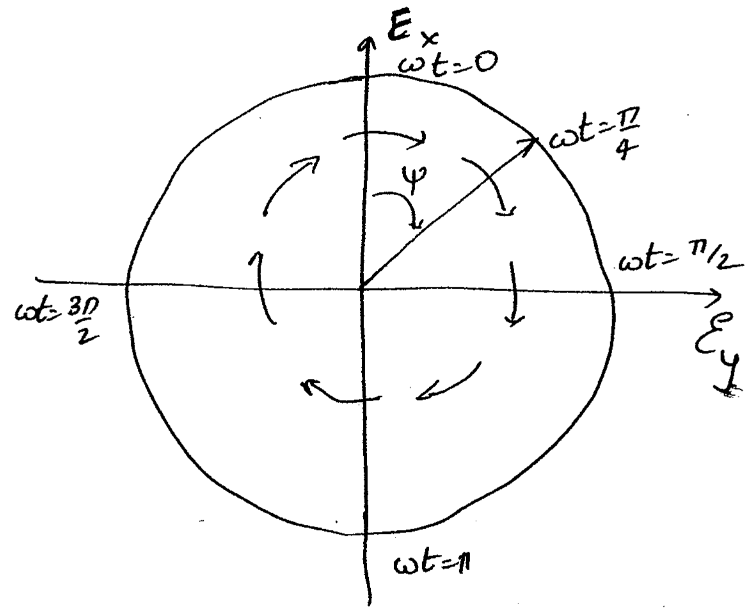
The amplitude of electric field is given by:

$$\mathcal{E} = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2} = \sqrt{E_R^2(\cos^2\omega t + \sin^2\omega t)} = E_R$$

and is directed along a line making an angle ψ with x axis, which is given by

$$\psi = \tan^{-1}\left[\frac{\mathcal{E}_y}{\mathcal{E}_x}\right] = \tan^{-1}\left[\frac{E_R \sin(\omega t)}{E_R \cos(\omega t)}\right] = \tan^{-1}\left[\tan(\omega t)\right]$$

$$= \omega t$$



If we plot locus of the electric field vector for various times at the $z=0$ plane, we see that it forms a circle of radius E_R and it rotates clockwise with an angular frequency of ω. Thus the wave is said to have a right hand circular polarization.

We can write the instantaneous electric field vector as

$$\mathcal{E} = Re\left[\hat{a}_x E_R e^{j(\omega t - \beta z)} + \hat{a}_y E_R e^{j(\omega t - \beta z - \pi/2)}\right]$$

$$= E_R \; Re\left\{\left[\hat{a}_x - j\hat{a}_y\right] e^{j(\omega t - \beta z)}\right\}$$

We note that there is a $90°$ phase difference between the two orthogonal components of the electric field vector

Left hand Circular Polarization

If the electric field vector has a counterclockwise sense of rotation, the polarization is designated as left-hand polarization.

Assume

$$\phi_x = 0 \qquad \phi_y = \frac{\pi}{2}$$

$$E_{x_0}^+ = E_{y_0}^+ = E_L$$

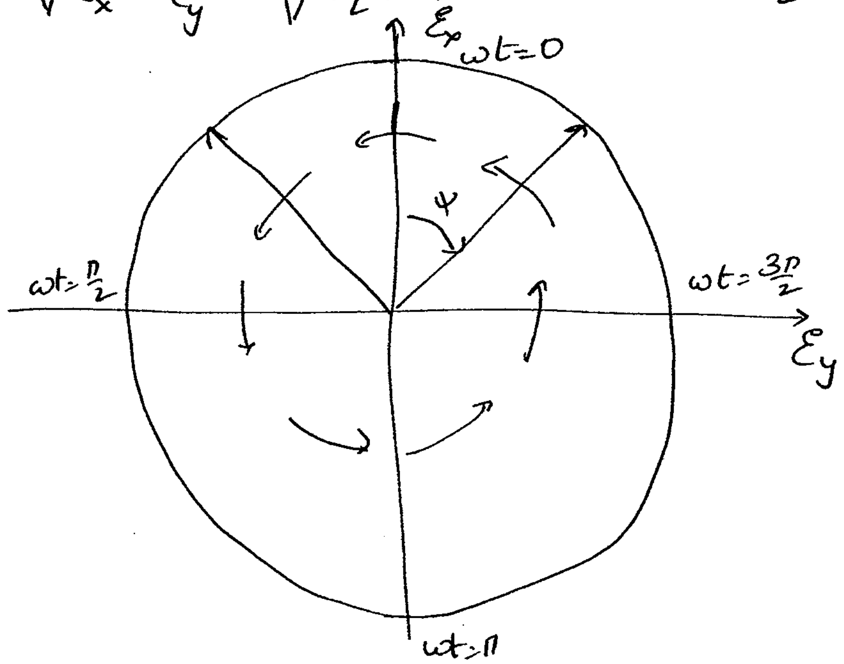
then

$$\mathcal{E}_x = E_L \cos(\omega t)$$

$$\mathcal{E}_y = E_L \cos\left(\omega t + \frac{\pi}{2}\right) = -E_L \sin(\omega t)$$

and the amplitude is

$$\mathcal{E} = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2} = \sqrt{E_L^2(\cos^2\omega t + \sin^2\omega t)} = E_L$$



The angle ψ is given by

$$\psi = \tan^{-1}\left[\frac{\mathcal{E}_y}{\mathcal{E}_x}\right] = \tan^{-1}\left[\frac{-E_L \sin(\omega t)}{E_L \cos(\omega t)}\right] = -\omega t$$

The locus of field vector is a circle of radius E_L and it rotates counter clockwise with an angular frequency 'ω'

hence it is left hand circular polarization

The instantaneous electric field vector can be written as

$$\mathcal{E} = Re\left[\hat{a}_x E_L e^{j(\omega t - \beta z)} + \hat{a}_y E_L e^{j(\omega t - \beta z + \pi/2)}\right]$$

$$= E_L \, Re\left[(\hat{a}_x + j\hat{a}_y)\, e^{j(\omega t - \beta z)}\right]$$

There is a 90° phase advance of $\mathcal{E}_y$ component relative to $\mathcal{E}_x$ component

The necessary and sufficient condition for circular polarization are:

1) The field must have two orthogonal linearly polarized components
2) The two components must have the same amplitude.
3) The two components must have a time phase difference of odd multiples of 90°

Elliptical polarization

A wave is said to be elliptically polarized if the tip of the electric field vector traces an elliptical locus in space. It is right hand elliptically polarized if the electric field vector rotates clockwise and is left hand elliptically polarized if electric field vector of the ellipse rotates counterclockwise

At $z=0$ plane

$$\phi_x = \frac{\pi}{2}$$

$$\phi_y = 0$$

$E_-^+ = (E_R + E_L)$ $\qquad E_+^+ (E_R - E_L)$

Then

$$\mathcal{E}_x = (E_R + E_L)\cos(\omega t + \tfrac{\pi}{2}) = -(E_R + E_L)\sin\omega t$$
$$\mathcal{E}_y = (E_R - E_L)\cos(\omega t)$$

The locus for the amplitude of electric field vector can be written as:

$$\mathcal{E}^2 = \mathcal{E}_x^2 + \mathcal{E}_y^2 = E_R^2 + E_L^2 + 2E_R E_L\,[\sin^2\omega t - \cos^2\omega t] \quad -(A)$$

However

$$\left.\begin{aligned} \sin\omega t &= -\mathcal{E}_x/(E_R+E_L) \\ \cos\omega t &= \mathcal{E}_y/(E_R-E_L) \end{aligned}\right\} \quad -(B)$$

Substituting (B) into (A)

$$\left\{\frac{\mathcal{E}_x}{E_R+E_L}\right\}^2 + \left\{\frac{\mathcal{E}_y}{E_R-E_L}\right\}^2 = 1$$

which is equation of an ellipse with major axis intercept $|\mathcal{E}|_{max} = |E_R + E_L|$ and minor axis intercept $|\mathcal{E}|_{min} = |E_R - E_L|$

The axial ratio

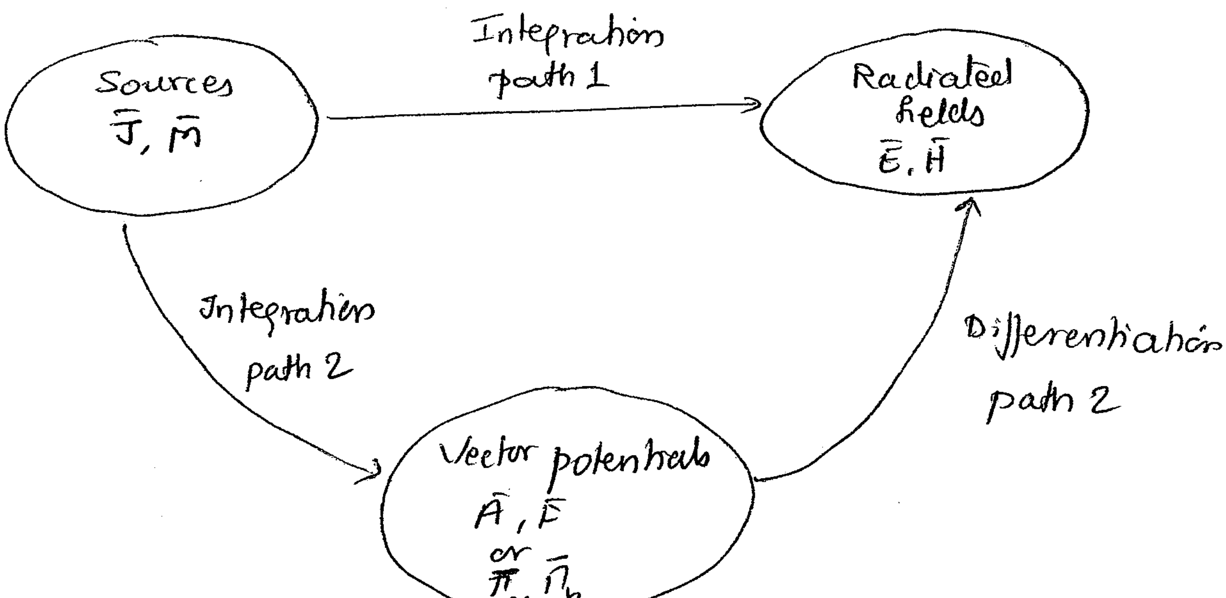
$$AR = -\frac{\mathcal{E}_{max}}{\mathcal{E}_{min}} = -\frac{(E_R+E_L)}{(E_R-E_L)}$$

The instantaneous electric field vector can be written as.

$$\mathcal{E} = Re\left\{[E_R(j\hat{a}_x + \hat{a}_y) + E_L(j\hat{a}_x - \hat{a}_y)]\,e^{j(\omega t - \beta z)}\right\}$$

Auxiliary vector Potential

The most common vector potential functions are the $\bar{A}$, magnetic vector potential, and $\bar{F}$, electric vector potential. These are used extensively in the solution of antenna radiation problems. The Hertz vector potential $\bar{\Pi}_e$ and $\bar{\Pi}_h$ make up another pair. The Hertz vector potential $\bar{\Pi}_e$ is analogous to $\bar{A}$ and $\bar{\Pi}_h$ is analogous to $\bar{F}$. The introduction of the potentials often simplifies the solution even though it may require determination of additional functions. While we can find $\bar{E}$ & $\bar{H}$ directly from the source current densities $\bar{J}$ and $\bar{M}$, it is usually simpler to find the auxiliary potential functions first and then determine $\bar{E}$ and $\bar{H}$



In a homogeneous medium, any solution for the time-harmonic electric and magnetic fields must satisfy the Maxwell's equations.

$$\nabla \times \bar{E} = -\bar{M} - j\omega\mu\bar{H} \quad - (1)$$

$$\nabla \times \bar{H} = \bar{J} + j\omega\varepsilon\bar{E} \quad - (2)$$

$$\nabla \cdot \bar{E} = \frac{q_{ev}}{\varepsilon} = \frac{\rho_v}{\varepsilon} \quad - (3)$$

$$\nabla \cdot \bar{H} = \frac{q_{mv}}{\mu} \quad - (4)$$

or the wave equations.

$$\nabla^2\bar{E} + \beta^2\bar{E} = \nabla \times \bar{M} + j\omega\mu\bar{J} + \frac{1}{\varepsilon}\nabla q_{ev}$$

$$\nabla^2\bar{H} + \beta^2\bar{H} = -\nabla \times \bar{J} + j\omega\varepsilon\bar{M} + \frac{1}{\mu}\nabla q_{mv}$$

where

$$\beta^2 = \omega^2\mu\varepsilon \qquad \text{(Lossless medium)}$$

For a source free region $\bar{J} = \bar{M} = q_{ev} = q_{mv} = 0$

$\bar{J} \rightarrow$ represents either actual or equivalent sources

$\bar{M} \rightarrow$ equivalent sources.

The vector Potential $\bar{A}$

From Gauss law for magnetostatics for a source free region

$$\nabla \cdot \bar{B} = 0 \quad - (5)$$

From vector identity

$$\nabla \cdot (\nabla \times \bar{A}) = 0 \quad — (6)$$

Comparing (5) and (6) we get

$$\bar{B} = \mu \bar{H} = \nabla \times \bar{A} \quad — (7)$$

$$\bar{H} = \frac{1}{\mu} \nabla \times \bar{A} \quad — (8)$$

Substituting into equation (1)

$$\nabla \times \bar{E} = -j\omega\mu \bar{H} \quad — (9)$$

$$\nabla \times \bar{E} = -j\omega (\nabla \times \bar{A}) \quad — (10)$$

which can be written as,

$$\nabla \times \bar{E} + j\omega (\nabla \times \bar{A}) = 0$$

$$\nabla \times (\bar{E} + j\omega \bar{A}) = 0 \quad — (11)$$

From another vector identity

$$\nabla \times (-\nabla \phi_e) = 0 \quad — (12)$$

Comparing (11) and (12)

$$\bar{E} + j\omega \bar{A} = -\nabla \phi_e$$

$$\boxed{\bar{E} = -\nabla \phi_e - j\omega \bar{A}} \quad — (13)$$

ϕ_e represents an arbitrary electric scalar potential

Again

$$\nabla \times \bar{A} = \mu \bar{H} \qquad -(14)$$

Taking curl on both sides.

$$\nabla \times \nabla \times \bar{A} = (\nabla \times \mu \bar{H})$$

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \nabla \times (\mu \bar{H}) \qquad -(15)$$

$$\mu \nabla \times \bar{H} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} \qquad -(16)$$

From Maxwells 2nd eqn, we have

$$\nabla \times \bar{H} = \bar{J} + j\omega\varepsilon \bar{E} \qquad -(17)$$

equating equation (16) & (17) we get

$$\mu \bar{J} + j\omega\mu\varepsilon \bar{E} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 A \qquad -(18)$$

Substituting (13) into (18) we get

$$\mu \bar{J} + j\omega\mu\varepsilon(-\nabla\phi_e - j\omega \bar{A}) = \nabla(\nabla \cdot A) - \nabla^2 \bar{A}$$

$$\mu \bar{J} + j\omega\mu\varepsilon(-\nabla\phi_e) + \omega^2\mu\varepsilon \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$$

$$\nabla^2 \bar{A} + \beta^2 \bar{A} = -\mu \bar{J} + \nabla(\nabla \cdot \bar{A}) + \nabla(j\omega\mu\varepsilon\phi_e)$$

$$= -\mu \bar{J} + \nabla(\nabla \cdot \bar{A} + j\omega\mu\varepsilon\phi_e) \qquad -(19)$$

The divergence of $\bar{A}$ can be written as

$$\nabla \cdot \bar{A} = -j\omega\mu\varepsilon\phi_e \Rightarrow \phi_e = -\frac{1}{j\omega\mu\varepsilon} \nabla \cdot \bar{A} \qquad -(20)$$

which is known as Lorentz condition.

Substituting (20) into (19) we get

$$\nabla^2 \bar{A} + \beta^2 \bar{A} = -\mu \bar{J}$$

eqn (13) becomes

$$\bar{E} = -\nabla \phi_e - j\omega \bar{A}$$

$$\boxed{\bar{E} = -j\omega \bar{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\bar{A})} \quad -(21)$$

Once $\bar{A}$ is found, $\bar{H}$ can be found using (8) & $\bar{E}$ can be obtained using (21).

<u>The vector Potential $\bar{F}$</u>

In a source free region from Gauss law

$$\nabla\cdot\bar{D} = 0 \quad - (1)$$

Again using identity

$$\nabla\cdot(\nabla\times\bar{F}) = 0 \quad - (2)$$

Comparing (1) & (2) we get

$$\bar{D} = -\nabla\times\bar{F}$$

or

$$\bar{E} = -\frac{1}{\varepsilon}\nabla\times\bar{F} \quad - (3)$$

From Maxwell's 2nd equation

$$\nabla \times \bar{H} = j\omega\varepsilon \bar{E} \quad -(4)$$

Substituting (3) in (4) we get

$$\nabla \times (\bar{H} + j\omega \bar{F}) = 0 \quad -(5)$$

Again

$$\nabla \times (-\nabla \phi_m) = 0 \quad -(6)$$

Comparing (5) & (6) we get

$$\bar{H} + j\omega \bar{F} = -\nabla \phi_m$$

$$\boxed{\bar{H} = -\nabla \phi_m - j\omega \bar{F}} \quad -(7)$$

Taking curl of (3) we get

$$\nabla \times \bar{E} = -\frac{1}{\varepsilon} \nabla \times \nabla \times \bar{F} = -\frac{1}{\varepsilon}\left[\nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}\right] \quad -(8)$$

From Maxwell's 1st equation

$$\nabla \times \bar{E} = -\bar{M} - j\omega\mu \bar{H} \quad -(9)$$

Comparing (8) & (9) we get

$$\nabla^2 \bar{F} + j\omega\mu\varepsilon \bar{H} = \nabla(\nabla \cdot \bar{F}) - \varepsilon \bar{M} \quad -(10)$$

Substituting (7) in (10) we get

$$\nabla^2 \bar{F} + \beta^2 \bar{F} = -\varepsilon \bar{M} + \nabla(\nabla \cdot \bar{F} + j\omega\mu\varepsilon \phi_m) \quad -(11)$$

Letting $\nabla \cdot \bar{F} = -j\omega\mu\varepsilon \phi_m \implies \phi_m = -\frac{1}{j\omega\mu\varepsilon} \nabla \cdot \bar{F} \quad -(12)$

$$\nabla^2 \bar{F} + \beta^2 \bar{F} = -\varepsilon \bar{M} \qquad -(14)$$

$$\bar{H} = -j\omega \bar{F} - \frac{j}{\omega \mu \varepsilon} \nabla(\nabla \cdot \bar{F}) \qquad -(15)$$

SUMMARY

1. Specify the electromagnetic boundary-value problem, which may or may not contain any sources within its boundaries, and its desired field configurations (modes).

2. a. Solve for **A** using (6-16),

$$\boxed{\nabla^2\mathbf{A} + \beta^2\mathbf{A} = -\mu\mathbf{J}} \quad \text{where } \beta^2 = \omega^2\mu\varepsilon \tag{6-30}$$

Depending on the problem, solutions for **A** in rectangular, cylindrical, and spherical coordinate systems take the forms found in Sections 3.4.1A, 3.4.2, and 3.4.3.

b. Solve for **F** using (6-28),

$$\boxed{\nabla^2\mathbf{F} + \beta^2\mathbf{F} = -\varepsilon\mathbf{M}} \quad \text{where } \beta^2 = \omega^2\mu\varepsilon \tag{6-31}$$

Depending on the problem, solutions for **F** in rectangular, cylindrical, and spherical coordinate systems take the forms found in Sections 3.4.1A, 3.4.2, and 3.4.3.

3. a. Find $\mathbf{H}_A$ using (6-4a) and $\mathbf{E}_A$ using (6-17). $\mathbf{E}_A$ can also be found using (6-12) by letting $\mathbf{J} = 0$.

$$\boxed{\mathbf{H}_A = \frac{1}{\mu}\nabla \times \mathbf{A}} \tag{6-32a}$$

$$\boxed{\mathbf{E}_A = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{A})} \tag{6-32b}$$

or

$$\boxed{\mathbf{E}_A = \frac{1}{j\omega\varepsilon}\nabla \times \mathbf{H}_A} \tag{6-32c}$$

b. Find $\mathbf{E}_F$ using (6-19) and $\mathbf{H}_F$ using (6-29). $\mathbf{H}_F$ can also be found using (6-24) by letting $\mathbf{M} = 0$.

$$\boxed{\mathbf{E}_F = -\frac{1}{\varepsilon}\nabla \times \mathbf{F}} \tag{6-33a}$$

$$\boxed{\mathbf{H}_F = -j\omega\mathbf{F} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{F})} \tag{6-33b}$$

or

$$\boxed{\mathbf{H}_F = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E}_F} \tag{6-33c}$$

4. The total fields are then found by the superposition of those given in step 3, that is,

$$\boxed{\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{A}) - \frac{1}{\varepsilon}\nabla \times \mathbf{F}} \tag{6-34}$$

or

$$\boxed{\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F = \frac{1}{j\omega\varepsilon}\nabla \times \mathbf{H}_A - \frac{1}{\varepsilon}\nabla \times \mathbf{F}} \tag{6-34a}$$

and

$$\boxed{\mathbf{H} = \mathbf{H}_A + \mathbf{H}_F = \frac{1}{\mu}\nabla \times \mathbf{A} - j\omega\mathbf{F} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{F})} \tag{6-35}$$

or

$$\boxed{\mathbf{H} = \mathbf{H}_A + \mathbf{H}_F = \frac{1}{\mu}\nabla \times \mathbf{A} - \frac{1}{j\omega\mu}\nabla \times \mathbf{E}_F} \tag{6-35a}$$

Whether (6-32b) or (6-32c) is used to find $\mathbf{E}_A$ and (6-33b) or (6-33c) to find $\mathbf{H}_F$ depends largely on the nature of the problem. In many instances one may be more complex than the other. For computing radiation fields in the far zone, it will be second term in each expression becomes negligible in this region. The same solution should be obtained using either of the two choices in each case.