THE GRADIENT

Suppose that a given vector function \( \mathbf{F}(x, y, z) \) has an associated scalar function \( \phi(x, y, z) \) and that these relate so

\[
\begin{align*}
\mathbf{F}_x &= \frac{\partial \phi}{\partial x} \\
\mathbf{F}_y &= \frac{\partial \phi}{\partial y} \\
\mathbf{F}_z &= \frac{\partial \phi}{\partial z}
\end{align*}
\]

where

\[
\mathbf{F}(x, y, z) = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}
\]

Consider now the line integral of \( \mathbf{F} \cdot \hat{\mathbf{t}} \) as we did when talking about circulation:

\[
\int_C \mathbf{F} \cdot \hat{\mathbf{t}} \, dc
\]

where \( C \) is the path of integration, \( \hat{\mathbf{t}} \) is the tangent vector at each point of the curve.

\[
\hat{\mathbf{t}} = \frac{\partial x}{\partial c} \mathbf{i} + \frac{\partial y}{\partial c} \mathbf{j} + \frac{\partial z}{\partial c} \mathbf{k}
\]

Then, \( \mathbf{F} \cdot \hat{\mathbf{t}} \) can be expressed as

\[
\mathbf{F} \cdot \hat{\mathbf{t}} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial c} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial c} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial c} = \frac{\partial \phi}{\partial c}
\]

Consider integrating now \( \mathbf{F} \cdot \hat{\mathbf{t}} \) between two points of the curve \( C \), say, \( P_0 \) and \( P_1 \).

\[
\int_{C_0} \mathbf{F} \cdot \hat{\mathbf{t}} \, dc = \int_{C_1} \frac{\partial \phi}{\partial c} \, dc = \phi(P_1) - \phi(P_0)
\]

So we can see now that the value of this integral depends on the initial and final points, but not from the path, as \( dc \) was simplified.
If we consider $P_0$ as a reference point, we can also write

$$\mathbf{u}(x, y, z) = \int_{P_0}^{P_1} \mathbf{F} \cdot d\mathbf{c}$$

This is only valid because we know the result is path-independent.

In general, to evaluate this integral, we can choose any arbitrary path of integration, as we know that our choice will not affect the result, providing we start always from $P_0$ and finish in our point of interest.

We choose, then, a path that has one portion of it that reduces to a 1-D integral on the $x$ direction.

Our integral from $P_0$ to $P$ can then be evaluated as

$$\mathbf{u}(x, y, z) = \int_{P_0}^{P_1} \mathbf{F} \cdot d\mathbf{c} + \int_{P_1}^{P} F_x(x, y, z) dx$$

But

$$\int_{P_1}^{P} F_x(x, y, z) dx = F_x(x, y, z) \bigg|_{P_1}^{P}$$

Independent of $x$, so

$$\frac{\partial}{\partial x} = 0$$

and if we consider only the variation in $x$,

$$\frac{dy}{dx} = \frac{\partial}{\partial x} \int_{P_1}^{P} F_x(x, y, z) dx = F_x(x, y, z) \Rightarrow F_x = \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial x} \right)$$

Similarly, choosing the appropriate integration paths,

$$F_y = \frac{\partial}{\partial y} \left( \frac{\partial y}{\partial x} \right)$$

and

$$F_z = \frac{\partial}{\partial z} \left( \frac{\partial y}{\partial x} \right)$$
So, since

\[ \mathbf{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \]

we have

\[ \mathbf{F} = \frac{\partial \psi}{\partial x} \hat{x} + \frac{\partial \psi}{\partial y} \hat{y} + \frac{\partial \psi}{\partial z} \hat{z} = \left[ \frac{\partial \psi}{\partial x} \begin{array}{c} 1 \\ 0 \\ 0 \end{array} + \frac{\partial \psi}{\partial y} \begin{array}{c} 0 \\ 1 \\ 0 \end{array} + \frac{\partial \psi}{\partial z} \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] \hat{y} \]

\[ \nabla \psi \quad \text{"DEL PSI"} \]

which is the gradient of \( \psi \) in Cartesian coordinates, which is a vector function \( \mathbf{F} \).

We can see that there is a relation between path independence and the existence of a scalar function

\[ \psi(x, y, z) \]

such that

\[ \mathbf{F} = \nabla \psi \]

If we consider a simply connected region, it can be demonstrated that if

\[ \nabla \times \mathbf{F} = 0 \]

then there exist a scalar function \( \psi(x, y, z) \) such that

\[ \mathbf{F} = \nabla \psi \]

We can use this knowledge to determine the electrostatic potential associated with an \( \mathbf{E} \) field. Since

\[ \nabla \times \mathbf{E} = 0 \]

we can say that

\[ \mathbf{E} = -\nabla \psi \]
WHERE SINCE \( \mathbf{D} \mathbf{V} \) IS A VECTOR IN THE DIRECTION OF INCREASING \( V \), THE FORCE ON A POSITIVE CHARGE \( q \) IS

\[
\mathbf{F} = q \mathbf{E} = -q \mathbf{D} \mathbf{V}
\]

WHICH IS IN THE DIRECTION OF DECREASING \( V \). THE NEGATIVE SIGN ENSURES THAT A POSITIVE CHARGE MOVES "DOWNHILL" FROM A HIGHER TO A LOWER POTENTIAL.

\[ \text{LA PLACIAN, LAPLACIAN EQUATION, POISSON EQUATION} \]

COMBINING

\[
\nabla \cdot \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} \quad \text{AND} \quad \mathbf{E} = -\nabla \mathbf{V}
\]

WE OBTAIN

\[
\nabla \cdot (\nabla \mathbf{V}) = -\frac{\mathbf{D}}{\varepsilon_0}
\]

EXPANDING THE LEFT-HAND SIDE,

\[
\nabla \cdot (\nabla \mathbf{V}) = \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \left[ \frac{\partial^2 \mathbf{V}}{\partial x^2} + \frac{\partial^2 \mathbf{V}}{\partial y^2} + \frac{\partial^2 \mathbf{V}}{\partial z^2} \right]
\]

\[
\nabla \cdot (\nabla \mathbf{V}) = \frac{\partial^2 \mathbf{V}}{\partial x^2} + \frac{\partial^2 \mathbf{V}}{\partial y^2} + \frac{\partial^2 \mathbf{V}}{\partial z^2}
\]

WE CAN WRITE THIS IN A MORE COMPACT WAY BY INTRODUCING A NEW OPERATOR CALLED "LA PLACIAN", DENOTED AS \( \nabla^2 \) (READ "DEL SQUARED")

\[
\nabla^2 = \nabla \cdot \nabla = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

THEN, \( \square \) CAN BE WRITTEN AS

\[
\nabla^2 \mathbf{V} = -\frac{\mathbf{D}}{\varepsilon_0} \quad \text{POISSON EQUATION,}
\]
A general definition of the Laplacian operator is

\[ \nabla^2 F = \nabla \cdot (\nabla F) \]

Definition of Laplacian operator.

If we consider the Poisson equation in a medium with no charges,

\[ \phi = 0 \implies \nabla^2 \phi = 0 \]

Laplace equation.

This expression is known as the Laplace equation.