NAME:

ECE 6540 FINAL

Show your work. Closed book, limited notes (1 page). No laptops or calculators.

1. Consider the DC value in noise problem: $x[n] = A + w[n]$ where w is WGN with variance σ^2 and A is Gaussian with variance σ_A^2. The Bayesian mean square error for the MMSE estimator given N observations is

$$\frac{1}{\frac{1}{\sigma_A^2} + \frac{N}{\sigma^2}}$$

   (a) Consider the following two scenarios:

      (i) $\sigma_A^2 = 1$ and $\sigma^2 = 10$.

      (ii) $\sigma_A^2 = 10$ and $\sigma^2 = 1$.

      Which of these scenarios result in a more accurate estimator when there is a larger number of observations $x[n]$ available? (in fact $N > 1$ is enough to give the same result) Give a brief intuitive explanation for the reason.

   (b) Assume that the prior PDF for A is an uniform distribution $\mathcal{U}[-\alpha/2, \alpha/2]$. Give a condition for α which reduces the maximum aposteriori (MAP) estimation for A to maximum likelihood estimation (MLE). Briefly explain why. You don't need to prove your answer.

2. Let the parameter θ and the data x be jointly Gaussian:

$$\begin{bmatrix} \theta \\ x \end{bmatrix} \sim \mathcal{N}(\mu, \mathbf{C})$$

where the mean is

$$\mu = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the covariance is

$$\mathbf{C} = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 4 \end{bmatrix}.$$

We observe $x = 3$.

(a) Find the value of the MMSE and MAP estimators for θ.

(b) If the covariance is matrix is changed to

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix},$$

what is the new MMSE estimator for the observation $x = 3$?

3. Consider the DC value in noise problem: $x[n] = A + w[n]$ where $w[n]$ is WGN with known variance σ^2. The prior PDF for A is given as

$$p(A) = 0.5\delta(A+1) + 0.5\delta(A-1).$$

In other words, A can take on the value -1 with probability 0.5 and the value 1 with probability 0.5. Let $\mathbf{x} = [x[0] \ldots x[N-1]]^T$ be the vector of observations.

(a) Find the MAP estimator in terms of $x[n]$ only.

(b) Assume that we collect the data set $y[n] = w[n]$ where $w[n]$ is WGN. Notice that $y[n]$ are independent of A. What is the MMSE estimator for A based on $y[n]$? Use the same prior PDF as above.

4. Consider the model $x[n] = \theta n + w[n]$ where $w[n]$ is WGN with variance σ^2 and $w[n]$ are independent of θ. Notice that the data model here is a line with slope θ passing through the origin. We also know that $E[\theta] = 0$ and $var(\theta) = \sigma_\theta^2$. Let $\hat{\theta}[N]$ denote the LMMSE estimator for θ from the observations $x[0], \ldots x[N]$.

   (a) Find the LMMSE estimator $\hat{\theta}[0]$.

   Hint: Recall that the LMMSE estimator $\hat{\theta}[0]$ can be found by projecting θ onto the subspace spanned by the data:

$$\frac{< \theta, x[0] >}{< x[0], x[0] >^2} x[0],$$

   where the inner products are defined as $< y, z >= E[yz]$

   (b) Find $\tilde{x}[1]$ (the portion of $x[1]$ orthogonal to $x[0]$) and find the LMMSE estimator $\hat{\theta}[1]$ using the additivity property of LMMSE estimation for orthogonal data sets.

   (c) The Bayesian mean square error (Bmse) for $\hat{\theta}[N]$ is given as

$$Bmse(\hat{\theta}[N]) = \frac{1}{\frac{1}{\sigma_\theta^2} + \frac{\sum_{n=0}^{N} n^2}{\sigma^2}}$$

   Recall the DC value problem $x[n] = A + w[n]$. Define $\hat{A}[N]$ to be the LMMSE estimator for A from the observations $x[0], \ldots x[N]$. The Bmse in this case is

$$Bmse(\hat{A}[N]) = \frac{1}{\frac{1}{\sigma_A^2} + \frac{N+1}{\sigma^2}}.$$

   If $\sigma_A^2 = \sigma_\theta^2$, we find that $Bmse(\hat{\theta}[N]) < Bmse(\hat{A}[N])$ for $N > 1$. Briefly give an intuitive reason why this is the case.