NAME:

ECE 6540 MIDTERM 2

Show your work. Closed book, limited notes (1 page). No laptops or calculators.

1. Consider the data model $x[n] = \theta s[n] + w[n]$ where $s[n]$ is a known signal and θ is an unknown scalar parameter. We have N observations: $\mathbf{x} = (x[0]\ x[1]\ \ldots\ x[N-1])^T$.

   Let $\mathbf{w} = (w[0]\ w[1]\ \ldots\ w[N-1])^T$ be the noise vector and let $\mathbf{C}$ denote the $N \times N$ covariance matrix for $\mathbf{w}$.

   An estimator for θ is said to be linear in the data if it is of the form

   $$\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n] = \mathbf{a}^T \mathbf{x} \tag{1}$$

   where $\mathbf{a} = (a_0\ a_1\ \ldots\ a_{N-1})^T$ is at our choosing. Then the *BLUE* estimator chooses the vector $\mathbf{a}$ by minimizing

   $$\mathbf{a}^T \mathbf{C} \mathbf{a} \tag{2}$$

   subject to the constraint

   $$\mathbf{a}^T \mathbf{s} = 1 \tag{3}$$

   where $\mathbf{s} = (s[0]\ s[1]\ \ldots\ s[N-1])^T$.

   (a) The constraint given in equation (3) restricts *BLUE* to a certain class of estimators. What is that class of estimators? Back up your answer mathematically.

   (b) Show that equation (2) is the variance of $\hat{\theta}$.

2. Consider a binary random variable $y[n]$ which has the probability distribution

$$p(y[n]) = \begin{cases} \alpha, & y[n] = 0 \\ 1-\alpha, & y[n] = 1 \end{cases} \tag{1}$$

Given N observations $\mathbf{y} = (y[0]\ y[1]\ \dots\ y[N-1])^T$, let $k = \sum_{n=0}^{N-1} y[n]$. The joint probability distribution for the vector $\mathbf{y}$ is a Binomial distribution

$$p(\mathbf{y}; \alpha) = \frac{N!}{k!(N-k)!} \alpha^{N-k} (1-\alpha)^k$$

(a) Given N observations $\mathbf{y} = (y[0]\ y[1]\ \dots\ y[N-1])^T$, find the maximum likelihood estimator for α.

(b) Now define a new random variable $x[n]$ using $y[n]$. First $y[n]$ is drawn from the probability distribution given in equation (1). Then, if $y[n] = 0$ then $x[n]$ is chosen from a Normal distribution with mean 0 and variance A. If $y[n] = 1$ then $x[n]$ is chosen from a Normal distribution with mean 0 and variance B. What is the PDF for $x[n]$ and what is its variance?

3. Consider the data model $x[n] = A + w[n]$ where A is an unknown scalar parameter and w is WGN (white Gaussian noise) with known variance σ^2. Let $\alpha = A^2/\sigma^2$ denote the signal to noise ratio.

   In answering the following questions you may use the fact that the maximum likelihood estimator for A is $\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$ and its mean and variance are given as $E(\hat{A}) = A$ and $var(\hat{A}) = \sigma^2/N$.

   (a) Find the maximum likelihood estimator $\hat{\alpha}$ for the signal to noise ratio.

   (b) Show that this maximum likelihood estimator $\hat{\alpha}$ is biased.

   (c) What is the form of the asymptotic distribution for the maximum likelihood estimator $\hat{\alpha}$? What is the variance of the asymptotic distribution in terms of A, σ^2 and N?

4. Orthonormal basis are an important tool for signal analysis. Consider M basis functions and the signal model given as

$$s[n] = \sum_{m=1}^{M} \theta_m h_m[n]$$

where θ_m are unknown coefficients. Write the m'th basis function in vector form as

$$\mathbf{h}_m = [h_m[0] \; h_m[1] \; \ldots \; h_m[N-1])]^T.$$

For orthonormal basis, we have $\mathbf{h}_i^T \mathbf{h}_j = \delta_{ij}$. Also let $\mathbf{x} = [x[0] \; x[1] \; \ldots \; x[N-1]]^T$ be the observed data vector.

(a) The Haar wavelet functions form an orhonormal basis. For $N = 4$, the first 2 Haar basis vectors are defined as

$$\mathbf{h}_1 = \frac{1}{2}[1 \;\; 1 \;\; 1 \;\; 1]^T$$

and

$$\mathbf{h}_2 = \frac{1}{2}[1 \;\; 1 \; -1 \; -1]^T$$

Find the least squares estimators for the coefficients θ_1 and θ_2. Write the estimators as functions only of the data points $x[0], x[1], x[2]$ and $x[3]$.

(b) The reconstruction error is defined as

$$\epsilon = ||\mathbf{x} - \mathbf{H}\hat{\Theta}||^2$$

where

$$\mathbf{H} = [\mathbf{h}_1 \; \mathbf{h}_2 \; \ldots \; \mathbf{h}_M]$$

and $\hat{\Theta}$ is the least squares estimate for the coefficient vector. Show that for an orthonormal basis, the reconstruction error is equal to

$$\epsilon = \mathbf{x}^T(\mathbf{I} - \mathbf{H}\mathbf{H}^T)\mathbf{x}$$