① $\boxed{\text{Unbiased estimator}}$

a) $E(\hat{\theta}) = E(a^T x)$

$E(\hat{\theta}) = a^T E(x)$

$E(\hat{\theta}) = a^T \theta s = \theta \overset{1}{a^T s} = \theta$

b) $a^T C a = a^T E[w w^T] a$

$= a^T E[(x - E(x))(x - E(x))^T] a$

$= E[a^T (x - E(x))(x - E(x))^T a]$

$= E[(\underbrace{a^T x}_{\hat{\theta}} - \underbrace{a^T E(x)}_{E(\hat{\theta})})(a^T x - a^T E(x))$

$= E[(\hat{\theta} - E(\hat{\theta}))(\hat{\theta} - E(\hat{\theta}))^T]$

$= \mathrm{Var}(\hat{\theta})$

Note: The quantity inside the expectation is a scaler, hence we say $\mathrm{var}(\hat{\theta})$ instead of $C_{\hat{\theta}}$.

② a) $\ln p(y;\alpha) = \ln \frac{N!}{k!\,(N-k)!}$

$$+ (N-k)\ln\alpha$$

$$+ k \ln 1-\alpha$$

To find MLE for α:

$$\frac{\partial \ln p(y;\alpha)}{\partial\alpha} = \frac{N-k}{\alpha} - \frac{k}{1-\alpha} = 0$$

$$\frac{(N-k)(1-\alpha) - k\alpha}{\alpha(1-\alpha)} = 0$$

$$(N-k)(1-\alpha) - k\alpha = 0$$

$$N - N\alpha - k + \cancel{k\alpha} - \cancel{k\alpha} = 0$$

$$N - k = N\alpha \implies \hat{\alpha} = \frac{N-k}{N}$$

b) This is a mixture of 2 Gaussians:

$$p(x[n];\alpha) = \frac{\alpha}{\sqrt{2\pi A}} \exp\left[-\frac{x[n]^2}{2A}\right] + \frac{1-\alpha}{\sqrt{2\pi B}} \exp\left[-\frac{x[n]^2}{2B}\right]$$

$$\mathrm{Var}(x[n]) = \alpha A + (1-\alpha) B.$$

3) a) Since the MLE for the transformation of a parameter is the transformation of the MLE for the parameter:

$$\hat{\alpha}_{MLE} = \frac{(\hat{A}_{MLE})^2}{\sigma^2}$$

Now we know $\hat{A}_{MLE} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$

$$\text{so} \quad \hat{\alpha}_{MLE} = \frac{\left(\frac{1}{N} \sum_{n=0}^{N-1} x[n]\right)^2}{\sigma^2}$$

b) $$E[\hat{\alpha}_{MLE}] = \frac{1}{\sigma^2} E[(\hat{A}_{MLE})^2]$$

$$= \frac{1}{\sigma^2}\left(E[\hat{A}_{MLE}]^2 + var(\hat{A}_{MLE})\right)$$

$$= \frac{1}{\sigma^2}\left(A^2 + \frac{\sigma^2}{N}\right) = \frac{A^2}{\sigma^2} + \frac{1}{N}$$

$$\neq \frac{A^2}{\sigma^2}$$

c) $$\hat{\alpha}_{MLE} \overset{a}{\sim} \mathcal{N}\left(\frac{A^2}{\sigma^2}, I(\alpha)^{-1}\right)$$

$$\underset{\text{asymptotic}}{Var(\hat{\alpha}_{MLE})} = I(\alpha)^{-1} = \frac{(\partial \alpha / \partial A)^2}{-E\left[\frac{\partial^2 p(x; A)}{\partial A^2}\right]} = \left(\frac{\partial \alpha}{\partial A}\right)^2 I(A)^{-1}$$

$$= \left(\frac{2A}{\sigma^2}\right)^2 \frac{\sigma^2}{N} = \frac{4A^2}{\sigma^2 N}$$

(4) a) For an orthonormal basis, $H^T H = I$ therefore $\hat{\theta} = H^T x$. For the Haar basis with $N = 4$

$$\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = \begin{pmatrix} h_1^T \\ h_2^T \end{pmatrix} x = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{pmatrix}$$

So $\hat{\theta}_1 = \frac{1}{2} \sum_{n=0}^{3} x[n]$

$$\hat{\theta}_2 = \frac{1}{2} \left(x[0] + x[1] - x[2] - x[3] \right)$$

b) $$\varepsilon = \| x - H\hat{\theta} \|^2 = \| x - HH^T x \|^2 = \| (I - HH^T) x \|^2$$

$$= x^T (I - HH^T)(I - HH^T) x$$

$$= x^T \left[I - HH^T - HH^T + H \underbrace{H^T H}_{I} H^T \right] x$$

$$= x^T \left[I - 2HH^T + HH^T \right] x$$

$$= x^T (I - HH^T) x$$