(1) $x[n] = R + w[n] \quad n = 0, \ldots N-1$

$w[n] \sim N(0,1)$ WGN, w & R independent

$R \sim N(100, 0.011)$ prior PDF

so $\mu_R = 100 \quad \sigma_R^2 = 0.011 \quad \sigma^2 = 1$

we know in this case (Example 10.1) $\quad Bmse(\hat{R}) = \dfrac{1}{N/\sigma^2 + 1/\sigma_R^2}$

We need

$$\frac{1}{N/\sigma^2 + 1/\sigma_R^2} \leq 0.1^2$$

$$\frac{\sigma^2 \sigma_R^2}{N\sigma_R^2 + \sigma^2} \leq 0.01$$

$$10\ \cancel{0.1} \times 0.011 \leq \cancel{0.01}(N \times 0.011 + 0.1)$$

$$\frac{0.11 - 0.1}{0.011} \leq N$$

$$9.09 \leq N$$

But N has to be an integer so we need $N \geq 10$

---

(2) $x[n] = A r^n + w[n] \quad n=0, \ldots N-1$

r known, w WGN with σ^2, $A \sim N(0, \sigma_A^2)$

$$\underline{x} = \underbrace{\begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^{N-1} \end{bmatrix}}_{H} A + \underline{w}$$

Notice $\mu_A = 0$

$C_A = \sigma_A^2$

$C_w = \sigma^2 I$

$$\hat{A}_{MMSE} = E[A|x] = \mu_A + \left(C_A^{-1} + H^T C_w^{-1} H\right)^{-1} H^T C_w^{-1} (x - H\mu_A)$$

So $\hat{A}_{MMSE} = 0 + \left(1/\sigma_A^2 + \frac{1}{\sigma^2} H^T H \right)^{-1} H^T \frac{1}{\sigma^2} (x - H0)$

$$H^T H = \sum_{n=0}^{N-1} r^{2n}$$ notice this is a scalar

$$H^T x = \sum_{n=0}^{N-1} r^n x[n]$$ 〃

Therefore $$\hat{A}_{MMSE} = \frac{1/\sigma^2 \sum_{n=0}^{N-1} r^n x[n]}{1/\sigma_A^2 + 1/\sigma^2 \sum_{n=0}^{N-1} r^{2n}}$$

$$Bmse(\hat{A}) = C_{A|x} = \left(C_A^{-1} + H^T C_w H \right)^{-1}$$

$$= \left(1/\sigma_A^2 + 1/\sigma^2 \sum_{n=0}^{N-1} r^{2n} \right)^{-1}$$

---

(3) $$p(x,y) = \frac{1}{2\pi\sqrt{|C|}} \exp\left[\frac{-[x \ y]\, C^{-1} \begin{bmatrix} x \\ y \end{bmatrix}}{2} \right]$$

since x & y 0 mean

When $x = x_0$ we have

$$g(y) = p(x_0, y) = \frac{1}{2\pi |C|^{1/2}} e^{-h(y)/2}$$

where $$h(y) = [x_0 \ y] \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y \end{bmatrix} \Big/ 1-\rho^2$$

$$= \frac{x_0^2 + y^2 - 2\rho x_0 y}{1-\rho^2}$$

To maximize $g(y)$ minimize $h(y)$

$$\frac{\partial h}{\partial y} = \frac{2y - 2\rho x_0}{1-\rho^2} = 0$$

a) Therefore $y = \rho x_0$.

For jointly Gaussian random variables

$$E[y|x] = E[y] + \frac{\mathrm{cov}(x,y)}{\mathrm{var}(x)}(x - E[x])$$

substituting for this problem

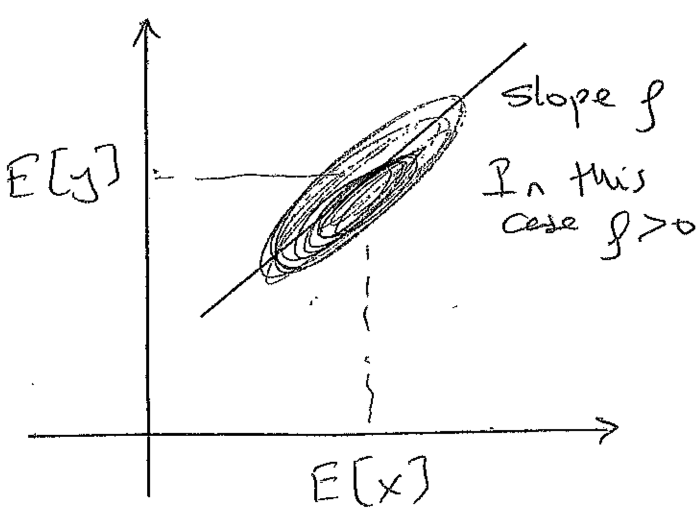
$$E[y|x] = 0 + \frac{\rho}{1}(x - 0) = \rho x$$

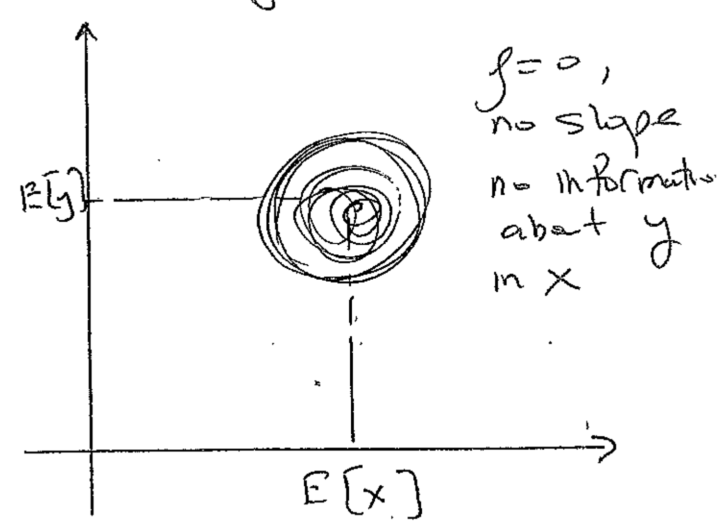
variance (x) from C

b) Therefore $E[y|x = x_0] = \rho x_0$.

c) The mode & mean are the same because $p(y|x)$ is a Gaussian.

d) $\rho = 0 \Rightarrow E[y|x] = E[y] = 0$





④ **11.2 Gaussian Mixture**

$$p(\theta|x) = \varepsilon N(x,1) + (1-\varepsilon) N(-x,1)$$

MSE: $E[\theta|x] = \varepsilon x + (1-\varepsilon)(-x)$

so $\varepsilon = 0.5$ $\quad E[\theta|x] = 0$

$\varepsilon = 3/4$ $\quad E[\theta|x] = \frac{3x}{4} - \frac{x}{4} = x/2$

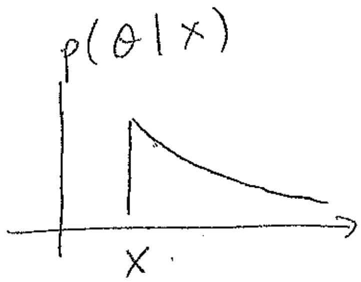
MAP For $\varepsilon = 0.5$ MAP is not unique : either x or $-x$

For $\varepsilon = 3/4$ MAP is x

11.3 exponential PDF

$$p(\theta|x) = \begin{cases} e^{-(\theta - x)}, & \theta > x \\ 0, & \theta < x \end{cases}$$

MMSE $E[\theta|x] = \int_x^\infty \theta e^{-(\theta-x)} d\theta$



$$= e^x \left[-\theta e^{-\theta} - e^{-\theta} \right]\Big|_x^\infty$$

$$= e^x \left(x e^{-x} + e^{-x} \right) = x + 1$$

MAP estimator is simply x

(5) $p_X(x[0], x[1] \mid A) = p_w(x[0]-A, x[1]-A \mid A)$

$= p_w(x[0]-A, x[1]-A)$ since w and A are independent

$= p_w(x[0]-A)\, p_w(x[1]-A)$ since $w[0]$ & $w[1]$ are independent

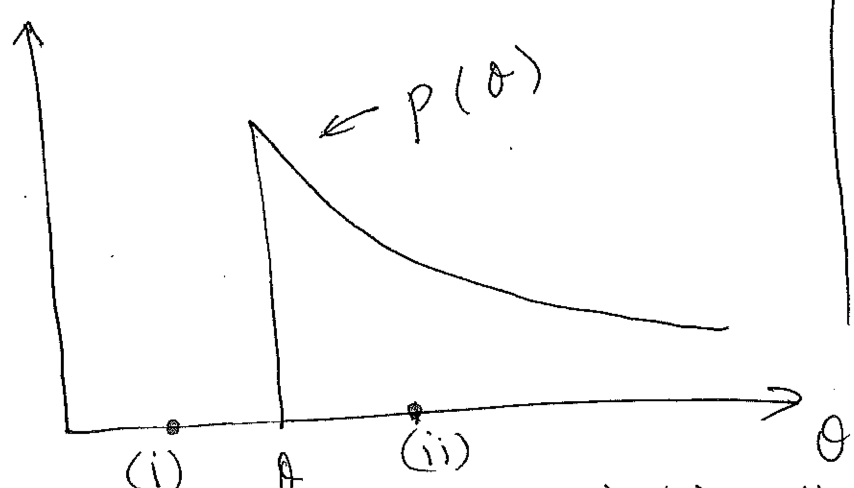
$= p_X(x[0] \mid A)\, p_X(x[1] \mid A)$ ✓

(6) a) $p(\underline{x} \mid \theta) = \begin{cases} (1/\theta)^N & 0 \le x[n] \le \theta \text{ for all } n \\ 0, & \text{otherwise} \end{cases}$

using conditional indep.

Rewrite as $p(\underline{x} \mid \theta) = \begin{cases} (1/\theta)^N, & \min x[n] \ge 0,\ \max x[n] \le \theta \\ 0, & \text{otherwise} \end{cases}$

b) $\hat{\theta}_{MAP} = \arg\max \underbrace{p(x \mid \theta)\, p(\theta)}_{g(\theta)}$



Combining the 2 cases:
$\hat{\theta}_{MAP} = \max(\theta_0, \max x[n])$

(i) $\max x[n] \le \theta_0$ $p(x|\theta)p(\theta)$ will be non-zero first at $\theta = \theta_0$ & since $p(\underline{x}|\theta)$ monotonically decreases with θ we must have $\hat{\theta}_{MAP} = \theta_0$

(ii) $\max x[n] > \theta_0$ $p(x|\theta)p(\theta)$ will be first non-zero at $\theta = \max x[n]$ therefore $\hat{\theta}_{MAP} = \max x[n]$